Poisson Homology in Degree 0 for some Rings of Symplectic Invariants

Frédéric Butin

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Abstract

Let $\mathfrak{g}$ be a finite-dimensional semi-simple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$, and W its Weyl group. The group W acts diagonally on $V := \mathfrak{h} \oplus \mathfrak{h}^*$, as well as on $\mathbb{C}[V]$. The purpose of this article is to study the Poisson homology of the algebra of invariants $\mathbb{C}[V]^W$ endowed with the standard symplectic bracket.

To begin with, we give general results about the Poisson homology space in degree 0, denoted by $HP_0(\mathbb{C}[V]^W)$, in the case where $\mathfrak{g}$ is of type $B_n - C_n$ or D_n, results which support Alev's conjecture. Then we are focusing the interest on the particular cases of ranks 2 and 3, by computing the Poisson homology space in degree 0 in the cases where $\mathfrak{g}$ is of type B_2 ($\mathfrak{so}_5$), D_2 ($\mathfrak{so}_4$), then B_3 ($\mathfrak{so}_7$), and $D_3 = A_3$ ($\mathfrak{so}_6 \simeq \mathfrak{sl}_4$). In order to do this, we make use of a functional equation introduced by Y. Berest, P. Etingof and V. Ginzburg. We recover, by a different method, the result established by J. Alev and L. Foissy, according to which the dimension of $HP_0(\mathbb{C}[V]^W)$ equals 2 for B_2. Then we calculate the dimension of this space and we show that it is equal to 1 for D_2. We also calculate it for the rank 3 cases, we show that it is equal to 3 for $B_3 - C_3$ and 1 for $D_3 = A_3$.

Key-words

Alev's conjecture ; Pfaff ; Poisson homology ; Weyl group ; invariants ; Berest-Etingof-Ginzburg equation.

1 Introduction

Let G be a finite subgroup of the symplectic group $\mathbf{Sp}(V)$, where V is a $\mathbb{C}-$vector space of dimension $2n$. Then the algebra of polynomial functions on V, denoted by $\mathbb{C}[V]$, is a Poisson algebra for the standard symplectic bracket, and as G is a subgroup of the symplectic group, the algebra of invariants, denoted by $\mathbb{C}[V]^G$, is also a Poisson algebra.

Several articles were devoted to the computation of Poisson homology and cohomology of the algebra of invariants $\mathbb{C}[V]^G$. In particular, Y. Berest, P. Etingof and V. Ginzburg, in [BEG04], prove that the $0-$th space of Poisson homology of $\mathbb{C}[V]^G$ is finite-dimensional.

After their works [AL98] and [AL98], J. Alev, M. A. Farinati, T. Lambre and A. L. Solotar establish a fundamental result in [AFLS00] : they compute all the spaces of Hochschild homology and cohomology of $A_n(\mathbb{C})^G$ for every finite subgroup G of $\mathbf{Sp}_{2n}\mathbb{C}$.

Besides, J. Alev and L. Foissy show in [AF06] that the dimension of the Poisson homology space in degree 0 of $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^W$ is equal to the one of the Hochschild homology space in degree 0 of $A_2(\mathbb{C})^W$, where $\mathfrak{h}$ is a Cartan subalgebra of a semi-simple Lie algebra of rank 2 with Weyl group W.

In the following, given a finite-dimensional semi-simple Lie algebra $\mathfrak{g}$, a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, and its Weyl group W, we are interested in the Poisson homology of $\mathbb{C}[V]^G$ in the case where V is the symplectic space $V := \mathfrak{h} \oplus \mathfrak{h}^*$ and $G := W$.

The group W acts diagonally on V, this induces an action of W on $\mathbb{C}[V]$. We denote by $\mathbb{C}[V]^W$ the algebra of invariants under this action. Endowed with the standard symplectic bracket, this algebra is a Poisson algebra. The action of the group W on V also induces an action of W on the Weyl algebra $A_n(\mathbb{C})$.

To begin with, we will give general results about the Poisson homology space in degree 0 of $\mathbb{C}[V]^W$ for the types $B_n - C_n$ and D_n, results which support Alev's conjecture and establish a framework for a possible proof. Sections 2.3, 2.4 and 2.5 contain the main results of this study.

Next we will use these results in order to completely calculate the Poisson homology space in degree 0, denoted by $HP_0(\mathbb{C}[V]^W)$, in the case where $\mathfrak{g}$ is $\mathfrak{so}_5$ (i.e. B_2) — so we recover, by a different method, the result established by J. Alev and L. Foissy for $\mathfrak{so}_5$ in the article [AF06], namely dim $HP_0(\mathbb{C}[V]^W) = 2$ — then in the case where $\mathfrak{g}$ is $\mathfrak{so}_4$ (i.e. $D_2 = A_1 \times A_1$) by showing that dim $HP_0(\mathbb{C}[V]^W) = 1$. Finally we will prove the important property for rank 3 :

Proposition 1 *(Poisson homology in degree $0-$ for $\mathfrak{g}$ of rank 3)*
Let $HP_0(\mathbb{C}[V]^W)$ be the Poisson homology space in degree 0 of $\mathbb{C}[V]^W$ and $HH_0\left(A_n(\mathbb{C})^W\right)$ the Hochschild homology space in degree 0 of $A_n(\mathbb{C})^W$.
For $\mathfrak{g}$ of type B_3 ($\mathfrak{so}_7$), we have $\dim HP_0\left(\mathbb{C}[V]^W\right) = \dim HH_0\left(A_n(\mathbb{C})^W\right) = 3$.
For $\mathfrak{g}$ of type $D_3 = A_3$ ($\mathfrak{so}_6 \simeq \mathfrak{sl}_4$), we have $\dim HP_0\left(\mathbb{C}[V]^W\right) = \dim HH_0\left(A_n(\mathbb{C})^W\right) = 1$.

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In order to carry out the computation we will use the functional equation given in the article [BEG04] quoted above.

2 Results about $B_n - C_n$ and D_n

As indicated above, we are interested in the Poisson homology of $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^W$, where $\mathfrak{h}$ is a Cartan subalgebra of a finite-dimensional semi-simple Lie algebra $\mathfrak{g}$, and W its Weyl group. We will study the types B_n and D_n. Recall that the root system of type C_n is dual to the root system of type B_n. So their Weyl groups are isomorphic, and the study of the case C_n is reduced to the study of the case B_n.

2.1 Definitions and notations

• Set $S := \mathbb{C}[\mathbf{x},\ \mathbf{y}] = \mathbb{C}[x_1, \dots,\ x_n,\ y_1, \dots,\ y_n]$.
For $m \in \mathbb{N}$, we denote by $S(m)$ the elements of S of degree m.
For B_n, we have $W = (\pm 1)^n \rtimes \mathfrak{S}_n = (\pm 1)^n \cdot \mathfrak{S}_n$ (permutations of the variables and sign changes of the variables).
For D_n, we have $W = (\pm 1)^{n-1} \rtimes \mathfrak{S}_n = (\pm 1)^{n-1} \cdot \mathfrak{S}_n$ (permutations of the variables and sign changes of an even number of variables).
Every element $(a_1,\ \dots,\ a_n) \in (\pm 1)^n$ is identified with the diagonal matrix $Diag(a_1,\ \dots,\ a_n)$, and every element $\sigma \in \mathfrak{S}_n$ is identified with the matrix $(\delta_{i,\sigma(j)})_{(i,j) \in [\![1,\ n]\!]}$. We will denote by s_j the $j-$th sign change, i.e.

$$s_j(x_k) = x_k \text{ if } k \neq j,\ s_j(x_j) = -x_j, \text{ and } s_j(y_k) = y_k \text{ if } k \neq j,\ s_j(y_j) = -y_j.$$

As for the elements of $(\pm 1)^{n-1}$, they are identified with the matrices of the form

$$Diag((-1)^{i_1},\ (-1)^{i_1+i_2},\ (-1)^{i_2+i_3},\ \dots,\ (-1)^{i_{n-2}+i_{n-1}},\ (-1)^{i_{n-1}}),$$

with $i_k \in \{0,\ 1\}$. We will denote by $s_{i,j}$ the sign change of the variables of indices i and j.
So, all these elements are in $\mathbf{O}_n\mathbb{C}$, and by identifying $g \in W$ with $\begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}$, we obtain $W \subset \mathbf{Sp}_{2n}\mathbb{C}$.

• The (right) action of W on S is defined for $P \in S$ and $g \in W$ by

$$g \cdot P(\mathbf{x},\ \mathbf{y}) := P\left(\sum_{j=1}^n g_{1j}x_j,\ \dots,\ \sum_{j=1}^n g_{nj}x_j,\ \sum_{j=1}^n g_{1j}y_j,\ \dots,\ \sum_{j=1}^n g_{nj}y_j\right).$$

So we have $h \cdot (g \cdot P) = (gh) \cdot P$.
In the particular case where $\sigma \in \mathfrak{S}_n$, we have $\sigma \cdot P(\mathbf{x},\ \mathbf{y}) = P(x_{\sigma^{-1}(1)},\ \dots,\ x_{\sigma^{-1}(n)},\ y_{\sigma^{-1}(1)},\ \dots,\ y_{\sigma^{-1}(n)})$.

• On S, we define the Poisson bracket

$$\{P,\ Q\} := \langle \nabla P,\ \nabla Q \rangle = \nabla P \cdot (J\, \nabla Q) = \nabla_{\mathbf{x}} P \cdot \nabla_{\mathbf{y}} Q - \nabla_{\mathbf{y}} P \cdot \nabla_{\mathbf{x}} Q,$$

where $\langle \cdot,\ \cdot \rangle$ is the standard symplectic product, associated to the matrix $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$.
As the group W is a subgroup of $\mathbf{Sp}(\langle \cdot,\ \cdot \rangle) = \mathbf{Sp}_{2n}\mathbb{C}$, the algebra of invariants S^W is a Poisson algebra for the bracket defined above.

• We define the Reynolds operator as the linear map R_n from S to S determined by

$$R_n(P) = \frac{1}{|W|} \sum_{g \in W} g \cdot P.$$

We set $A = \mathbb{C}[\mathbf{z},\ \mathbf{t}] = \mathbb{C}[z_1, \dots,\ z_n,\ t_1, \dots,\ t_n]$ and $S' := A[\mathbf{x},\ \mathbf{y}]$, and we extend the map R_n as a $A-$linear map from S' to S'.

Remark 2
In the case of B_n, every element of S^W has an even degree. (It is false for D_n).

2.2 Poisson homology

• Let A be a Poisson algebra. We denote by $\Omega^p(A)$ the $A-$module of Kähler differentials, i.e. the vector space spanned by the elements of the form $F_0\, dF_1 \wedge \dots \wedge dF_p$, where the F_j belong to A, and $d : \Omega^p(A) \to \Omega^{p+1}(A)$ is

the De Rham differential.
We consider the complex

$$\cdots \xrightarrow{\partial_5} \Omega^4(A) \xrightarrow{\partial_4} \Omega^3(A) \xrightarrow{\partial_3} \Omega^2(A) \xrightarrow{\partial_2} \Omega^1(A) \xrightarrow{\partial_1} \Omega^0(A)$$

with Brylinsky-Koszul boundary operator ∂_p (See [B88]) :

$$\begin{aligned}\partial_p(F_0\, dF_1 \wedge \cdots \wedge dF_p) &= \sum_{j=1}^{p}(-1)^{j+1}\{F_0,\, F_j\}\, dF_1 \wedge \cdots \wedge \widehat{dF_j} \wedge \cdots \wedge dF_p \\ &+ \sum_{1\le i<j\le p} (-1)^{i+j} F_0\, d\{F_i,\, F_j\} \wedge dF_1 \wedge \cdots \wedge \widehat{dF_i} \wedge \cdots \wedge \widehat{dF_j} \wedge \cdots \wedge dF_p.\end{aligned}$$

The Poisson homology space in degree p is given by the formula

$$HP_p(A) = \mathrm{Ker}\,\partial_p \,/\, \mathrm{Im}\,\partial_{p+1}.$$

In particular, we have $HP_0(A) = A \,/\, \{A,\, A\}$.

• In the sequel, we will denote $HP_0(\mathbb{C}[V]^W)$ by $HP_0(W)$ and $HH_0(\mathbb{C}[V]^W)$ by $HH_0(W)$.

2.3 Vectors of highest weight 0

The aim of this section is to show that $S^W(2)$ is isomorphic to $\mathfrak{sl}_2$ and that the vectors which do not belong to $\{S^W,\, S^W\}$, are among the vectors of highest weight 0 of the $\mathfrak{sl}_2-$module S^W, an observation which will simplify the calculations.

Proposition 3
For B_n $(n \geq 2)$ and D_n $(n \geq 3)$, the subspace $S^W(2)$ is isomorphic to $\mathfrak{sl}_2$. More precisely $S^W(2) = \langle E,\ F,\ H\rangle$ with the relations $\{H,\ E\} = 2E$, $\{H,\ F\} = -2F$ and $\{E,\ F\} = H$, where the elements $E,\ F,\ H$ are explicitly given by

$$E = \tfrac{n}{2}R_n(x_1^2) = \tfrac{1}{2}\,\mathbf{x}\cdot\mathbf{x} \quad F = -\tfrac{n}{2}R_n(y_1^2) = -\tfrac{1}{2}\,\mathbf{y}\cdot\mathbf{y} \quad H = -n\,R_n(x_1y_1) = -\mathbf{x}\cdot\mathbf{y}.$$

For D_2, we have $S^W(2) = \langle E,\, F,\, H\rangle \oplus \langle E',\, F',\, H'\rangle$ (direct sum of two Lie algebras isomorphic to $\mathfrak{sl}_2$).
So the spaces S and S^W are $\mathfrak{sl}_2-$modules.

Proof :
We demonstrate the proposition for B_n, the proof being analogous for D_n.
• As x_1^2 is invariant under sign changes, we may write $R_n(x_1^2) = \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n}\sigma\cdot x_1^2$. Moreover, we have the partition $\mathfrak{S}_n = \coprod_{j=1}^n A_j$, where $A_j := \{\sigma \in \mathfrak{S}_n \ /\ \sigma(1) = j\}$ has cardinality $(n-1)!$.
Thus $R_n(x_1^2) = \frac{(n-1)!}{n!}\sum_{j=1}^n x_j^2$. We proceed likewise with $R_n(y_1^2)$ and $R_n(x_1y_1)$.
• We obviously have $S^W(2) \supset \langle E,\ F,\ H\rangle$.
Moreover, $R(x_j^2) = R(x_1^2)$, $R(y_j^2) = R(y_1^2)$ and $R(x_jy_j) = R(x_1y_1)$. Last, if $i \neq j$, then x_ix_j, y_iy_j and x_iy_j are mapped to their opposite by the $i-$th sign change s_i, and

$$W = s_i\cdot\langle s_1,\ \ldots,\ s_{i-1},\ s_{i+1},\ \ldots,\ s_n\rangle\cdot\mathfrak{S}_n \sqcup \langle s_1,\ \ldots,\ s_{i-1},\ s_{i+1},\ \ldots,\ s_n\rangle\cdot\mathfrak{S}_n,$$

thus $R_n(x_ix_j) = R_n(y_iy_j) = R_n(x_iy_j) = 0$. Hence $S^W(2) \subset \langle E,\ F,\ H\rangle$.
• We have $\nabla E = \begin{pmatrix}\mathbf{x}\\ \mathbf{0}\end{pmatrix}$, $\nabla F = \begin{pmatrix}\mathbf{0}\\ -\mathbf{y}\end{pmatrix}$ and $\nabla H = \begin{pmatrix}-\mathbf{y}\\ -\mathbf{x}\end{pmatrix}$,
so $\{E,\ F\} = H$, $\{H,\ E\} = 2E$ and $\{H,\ F\} = -2F$.■

We denote by $S_{\mathfrak{sl}_2}$ the set of vectors of highest weight 0, i.e. the set of elements of S which are annihilated by the action of $\mathfrak{sl}_2$.
For $j \in \mathbb{N}$, we denote by $S(j)_{\mathfrak{sl}_2}$ the elements of $S_{\mathfrak{sl}_2}$ of degree j.
As the action of W and the action of $\mathfrak{sl}_2$ commute, we may write $(S^W)_{\mathfrak{sl}_2} = (S_{\mathfrak{sl}_2})^W = S^W_{\mathfrak{sl}_2}$ and likewise for $S^W(j)_{\mathfrak{sl}_2}$.

Proposition 4
• *If S^W contains no element of degree 1, then the vectors of highest weight 0 of degree 0 do not belong to $\{S^W,\ S^W\}$.*
• *Let W be of type B_n $(n \geq 2)$ or D_n $(n \geq 3)$. If S^W contains no element of degree 1, 3, , then the vectors of highest weight 0 of degree 4 do not belong to $\{S^W,\ S^W\}$.*

Proof :

• The Poisson bracket being homogeneous of degree -2, we have $S^W(0) \cap \{S^W,\ S^W\} = \{S^W(2),\ S^W(0)\} = \{0\}$.
• Similarly, $S^W(4) \cap \{S^W,\ S^W\} = \{S^W(0),\ S^W(6)\} + \{S^W(2),\ S^W(4)\} = \{\mathfrak{sl}_2,\ S^W(4)\}$, according to Proposition 3. With the decomposition of the $\mathfrak{sl}_2-$modules, we have $S^W(4) = \bigoplus_{m\in\mathbb{N}} V(m)$, with $\{\mathfrak{sl}_2,\ V(m)\} = V(m)$ if $m \in \mathbb{N}^*$ and $\{\mathfrak{sl}_2,\ V(0)\} = \{0\}$. So if $a \in S^W(4) \cap \{S^W,\ S^W\}$, then $a \in \bigoplus_{m\in\mathbb{N}^*} V(m)$.■

Proposition 5
For every monomial $M = \mathbf{x}^{\mathbf{i}}\mathbf{y}^{\mathbf{j}}$, we have

$$\begin{array}{rcl} \{H,\ M\} &=& (|\mathbf{i}| - |\mathbf{j}|)M = (\deg_x(M) - \deg_y(M))M \\ \{E,\ M\} &=& \sum_{k=1}^{n} j_k\, x_1^{i_1} \dots x_{k-1}^{i_{k-1}} x_k^{i_k+1} x_{k+1}^{i_{k+1}} \dots x_n^{i_n} y_1^{j_1} \dots y_{k-1}^{j_{k-1}} y_k^{j_k-1} y_{k+1}^{j_{k+1}} \dots y_n^{j_n}, \\ \{F,\ M\} &=& \sum_{k=1}^{n} i_k\, x_1^{i_1} \dots x_{k-1}^{i_{k-1}} x_k^{i_k-1} x_{k+1}^{i_{k+1}} \dots x_n^{i_n} y_1^{j_1} \dots y_{k-1}^{j_{k-1}} y_k^{j_k+1} y_{k+1}^{j_{k+1}} \dots y_n^{j_n}. \end{array}$$

In particular, every vector of highest weight 0 is of even degree.

Proof : This results from a simple calculation.

Remark 6
Let $j \in \mathbb{N}$ and $P \in S^W(j)$. According to the decomposition of $\mathfrak{sl}_2-$module $S^W(j)$ in weight subspaces, we may write $P = \sum_{k=-m}^{m} P_k$ with $\{H,\ P_k\} = k\,P_k$. So we have $\boxed{\frac{S^W(j)}{\{S^W,\ S^W\}\cap S^W(j)} = \frac{S^W(j)_{\mathfrak{sl}_2}}{\{S^W,\ S^W\}\cap S^W(j)_{\mathfrak{sl}_2}}}$.
Thus the vectors which do not belong to $\{S^W,\ S^W\}$ are to be found among the vectors of highest weight 0.

The following property is a generalization of Proposition 3 proved by J. Alev and L. Foissy in [AF06]. It enables us to know the Poincaré series of the algebra $S_{\mathfrak{sl}_2}$.

Proposition 7
For $l \in \mathbb{N}$, we have $S_{\mathfrak{sl}_2}(2l+1) = \{0\}$ and $\dim\ S_{\mathfrak{sl}_2}(2l) = (C_{l+n-1}^{n-1})^2 - C_{l+n}^{n-1} C_{l+n-2}^{n-1}$.

The following result is important for solving the equation of Berest-Etingof-Ginzburg, because it gives a description of the space of vectors of highest weight 0, space in which we will search the solutions of this equation. In the proof of this proposition, we use the articles [DCP76] and [GK04] concerning the pfaffian algebras.

Proposition 8
For $i \neq j$, set $X_{i,j} := x_i y_j - y_i x_j$. Then the algebra $\mathbb{C}[\mathbf{x},\ \mathbf{y}]_{\mathfrak{sl}_2}$ is the algebra generated by the $X_{i,j}$'s for $(i,\ j) \in [\![1,\ n]\!]^2$. We denote this algebra by $\mathbb{C}\langle X_{i,j}\rangle$.
This algebra is not a polynomial algebra for $n \geq 4$ (e. g. $X_{1,2}X_{3,4} - X_{1,3}X_{2,4} + X_{2,3}X_{1,4} = 0$).

Proof :

• The inclusion $\mathbb{C}\langle X_{i,j}\rangle \subset \mathbb{C}[\mathbf{x},\ \mathbf{y}]_{\mathfrak{sl}_2}$ being obvious, all that we have to do is to show that the Poincaré series of both spaces are equal, knowing that the one for $\mathbb{C}[\mathbf{x},\ \mathbf{y}]_{\mathfrak{sl}_2}$ is already given by Proposition 7.
• Consider the vectors $u_j := \begin{pmatrix} x_j \\ y_j \end{pmatrix}$ for $j = 1 \dots n$, in the symplectic space $\mathbb{C}^2$ endowed with the standard symplectic form $\langle\cdot\rangle$ defined by the matrix $J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Let $\{T_{i,j}\ /\ 1 \leq i < j \leq n\}$ be a set of indeterminates, and let $\widetilde{T}$ be the antisymmetric matrix the general term of which is $T_{i,j}$ if $i < j$. Then, according to section 6 of [DCP76], the ideal I_2 of relations between the $\langle u_i,\ u_j\rangle$ (i.e. between the $X_{i,j}$) is generated by the pfaffian minors of $\widetilde{T}$ of size 4×4. Set $PF := \mathbb{C}\langle X_{i,j}\rangle$. So we have $PF \simeq \mathbb{C}[(T_{i,j})_{i<j}]\,/\,I_2 =: PF_0$, i.e. the algebra PF is isomorphic to the pfaffian algebra PF_0. Its Poincaré series is given in section 4 of [GK04] by

$$\dim\ PF_0(m) = (C_{m+n-2}^{m})^2 - C_{m+n-2}^{m-1} C_{m+n-2}^{m+1}.$$

So we have $\dim\ PF(2l) = (C_{l+n-2}^{l})^2 - C_{l+n-2}^{l-1} C_{l+n-2}^{l+1}$. We verify that

$$\dim\ PF(2l) = (C_{l+n-1}^{n-1})^2 - C_{l+n}^{n-1} C_{l+n-2}^{n-1} = \dim\ \mathbb{C}[\mathbf{x},\ \mathbf{y}]_{\mathfrak{sl}_2}(2l).$$

We have obviously $\dim\ PF(2l+1) = 0 = \dim\ \mathbb{C}[\mathbf{x},\ \mathbf{y}]_{\mathfrak{sl}_2}(2l+1)$. Hence the equality of the Poincaré series.■

2.4 Equation of Berest-Etingof-Ginzburg

We study the functionnal equation introduced by Y. Berest, P. Etingof and V. Ginzburg in [BEG04]. The point is that solving this equation, in the space $S^W_{\mathfrak{sl}_2}$, is equivalent to the determination of the quotient $\frac{S^W}{\{S^W,\ S^W\}}$, that is to say the computation of the Poisson homology space in degree 0 of S^W.

Lemma 9 *(Berest - Etingof - Ginzburg)*
We consider $\mathbb{C}^{2n}$, endowed with its standard symplectic form, denoted by $\langle\cdot,\ \cdot\rangle$. Let $j \in \mathbb{N}$.
Let $S := \mathbb{C}[\mathbf{x}, \mathbf{y}] = \mathbb{C}[\mathbf{z}]$, and let $\mathcal{L}_j := \left(\frac{S^W(j)}{\{S^W,\ S^W\}\cap S^W(j)}\right)^*$ be the linear dual of $\frac{S^W(j)}{\{S^W,\ S^W\}\cap S^W(j)}$.
Then $\mathcal{L}_j$ is isomorphic to the vector space of polynomials $P \in \mathbb{C}[\mathbf{w}]^W(j)$ satisfying the following equation :

$$\forall\ \mathbf{w},\ \mathbf{w}' \in \mathbb{C}^{2n},\ \sum_{g\in W} \langle \mathbf{w},\ g\mathbf{w}'\rangle\ P(\mathbf{w}+g\mathbf{w}') = 0 \tag{1}$$

Proof :
• For $\mathbf{w} = (\mathbf{u},\ \mathbf{v}) \in \mathbb{C}^{2n}$ and $\mathbf{z} = (\mathbf{x},\ \mathbf{y}) \in \mathbb{C}^{2n}$, we set $L_{\mathbf{w}}(\mathbf{z}) := \sum_{g\in W} e^{\langle \mathbf{w},\ g\mathbf{z}\rangle}$.
So we have $\{L_{\mathbf{w}}(\mathbf{z}),\ L_{\mathbf{w}'}(\mathbf{z})\} = \nabla_{\mathbf{x}} L_{\mathbf{w}}(\mathbf{z}) \cdot \nabla_{\mathbf{y}} L_{\mathbf{w}'}(\mathbf{z}) - \nabla_{\mathbf{y}} L_{\mathbf{w}}(\mathbf{z}) \cdot \nabla_{\mathbf{x}} L_{\mathbf{w}'}(\mathbf{z})$

We deduce the formula

$$\{L_{\mathbf{w}}(\mathbf{z}),\ L_{\mathbf{w}'}(\mathbf{z})\} = \sum_{g\in W} \langle \mathbf{w},\ g\mathbf{w}'\rangle L_{\mathbf{w}+g\mathbf{w}'}(\mathbf{z}). \tag{2}$$

• Moreover, $L_{\mathbf{w}}(\mathbf{z})$ is a power series in $\mathbf{w}$, the coefficients of which generate S^W :

$$L_{\mathbf{w}}(\mathbf{z}) = \sum_{p=0}^{\infty} \frac{|W|}{p!} R_n \left[\left(\sum_{i=1}^{n} y_i u_i - x_i v_i \right)^p \right] \tag{3}$$

The coefficients of the series are the images by R_n of the elements of the canonical basis of S.

Remark : in the case of B_n, there is no invariant of odd degree, so we have $L_{\mathbf{w}}(\mathbf{z}) = \sum_{g\in W} \mathrm{ch}(\langle \mathbf{w},\ g\mathbf{z}\rangle)$.

$\triangleright$ Set $M_p(\mathbf{z}) = \{\mathbf{z}^{\mathbf{i}}\ /\ |\mathbf{i}| = p\}$ and $M_p(\mathbf{w}) = \{\mathbf{w}^{\mathbf{i}}\ /\ |\mathbf{i}| = p\}$.
For a monomial $m = \mathbf{x}^i\mathbf{y}^j \in M_p(\mathbf{z})$, let $\widetilde{m} = \mathbf{u}^j\mathbf{v}^i$.
Similarly, for a monomial $m = \mathbf{u}^i\mathbf{v}^j \in M_p(\mathbf{w})$, let $\overline{m} = \mathbf{x}^j\mathbf{y}^i$.
So, for $m \in M_p(\mathbf{z})$, we have $\overline{\widetilde{m}} = m$, and for $m \in M_p(\mathbf{w})$, we have $\widetilde{\overline{m}} = m$.

The series $L_{\mathbf{w}}(\mathbf{z})$ may then be written as

$$L_{\mathbf{w}}(\mathbf{z}) = |W| + \sum_{j=1}^{\infty} \sum_{m_j\in M_j(\mathbf{w})} \alpha_{m_j} R_n(\overline{m_j}) m_j = |W| + \sum_{j=1}^{\infty} \sum_{m_j\in M_j(\mathbf{z})} \alpha_{\widetilde{m_j}} R_n(m_j)\widetilde{m_j} = |W| + \sum_{j=1}^{\infty} L^j_{\mathbf{w}}(\mathbf{z}), \tag{4}$$

with $\alpha_{m_j} \in \mathbb{Q}^*$.
$\triangleright$ Now $\left(\sum_{i=1}^n y_iu_i - x_iv_i\right)^p = \sum_{|\mathbf{a}|+|\mathbf{b}|=p} (-1)^{|\mathbf{b}|} C_p^{\mathbf{a},\mathbf{b}} \mathbf{x}^{\mathbf{b}}\mathbf{y}^{\mathbf{a}}\mathbf{u}^{\mathbf{a}}\mathbf{v}^{\mathbf{b}}$, where $C_p^{\mathbf{a},\mathbf{b}} = \frac{p!}{a_1!\ldots a_n! b_1!\ldots b_n!}$ is the multinomial coefficient, therefore according to formula (3), we have

$$L_{\mathbf{w}}(\mathbf{z}) = |W| + \sum_{p=1}^{\infty} \sum_{|\mathbf{a}|+|\mathbf{b}|=p} (-1)^{|\mathbf{b}|} \frac{|W|}{p!} C_p^{\mathbf{a},\mathbf{b}} R_n\left(\mathbf{x}^{\mathbf{b}}\mathbf{y}^{\mathbf{a}}\right) \mathbf{u}^{\mathbf{a}}\mathbf{v}^{\mathbf{b}} \tag{5}$$

By collecting the formulae (4) and (5), we obtain

$$\alpha_{\mathbf{u}^{\mathbf{a}}\mathbf{v}^{\mathbf{b}}} = (-1)^{|\mathbf{b}|} \frac{|W|}{p!} C_p^{\mathbf{a},\mathbf{b}}. \tag{6}$$

• We identify $\mathcal{L}_j$ with the vector space of linear forms on $S^W(j)$ which vanish on $\{S^W,\ S^W\} \cap S^W(j)$.
Define the map

$$\begin{array}{rccl} \pi\ : & \mathcal{L}_j & \rightarrow & \{P \in \mathbb{C}[\mathbf{w}]^W(j)\ /\ \forall\ \mathbf{w},\ \mathbf{w}' \in \mathbb{C}^{2n},\ \sum_{g\in W} \langle \mathbf{w},\ g\mathbf{w}'\rangle\ P(\mathbf{w}+g\mathbf{w}') = 0\} \\ & f & \mapsto & \pi_f := f(L^j_{\mathbf{w}}). \end{array} \tag{7}$$

Then π is well defined : indeed $L^j_{\mathbf{w}}$ is a polynomial in $\mathbf{z}$ of degree j with coefficients in $\mathbb{C}[\mathbf{w}]$, and explicitly, we have

$$f(L^j_{\mathbf{w}}) = \sum_{m_j\in M_j(\mathbf{z})} \alpha_{\widetilde{m_j}} f(R_n(m_j))\widetilde{m_j} \in \mathbb{C}[\mathbf{w}]. \tag{8}$$

$\triangleright$ If two monomials $m_j,\ m'_j \in M_j(\mathbf{z})$ belong to a same orbit under the action of $\mathfrak{S}_n$, then the coefficients $\alpha_{\widetilde{m_j}} f(R_n(m_j))$ and $\alpha_{\widetilde{m'_j}} f(R_n(m'_j))$ of m_j and m'_j are the same, thus $f(L^j_{\mathbf{w}})$ is invariant under W.

$\triangleright$ Besides, π_f is solution of $E_n(P) = 0$: indeed, we may extend f as a linear map defined on $\frac{S^W}{\{S^W,\ S^W\}} = \bigoplus_{i=0}^{\infty} \frac{S^W(i)}{\{S^W,\ S^W\}\cap S^W(i)}$, by setting $f = 0$ on $\frac{S^W(i)}{\{S^W,\ S^W\}\cap S^W(i)}$ for $i \neq j$.
Then, according to (2), we have the equality

$$0 = f\left(\{L_{\mathbf{w}},\ L_{\mathbf{w}'}\}\right) = \sum_{g\in W} \langle \mathbf{w},\ g\mathbf{w}'\rangle f\left(L_{\mathbf{w}+g\mathbf{w}'}\right),$$

hence $\sum_{g\in W} \langle \mathbf{w},\ g\mathbf{w}'\rangle f\left(L^j_{\mathbf{w}+g\mathbf{w}'}\right) = 0$. So, the polynomial $f(L^j_{\mathbf{w}}) \in \mathbb{C}[\mathbf{w}]$ satisfies equation (1).

- Define the map

$$\begin{array}{rcl} \varphi\ :\ \{P \in \mathbb{C}[\mathbf{w}]^W(j)\ /\ \forall\ \mathbf{w},\ \mathbf{w}' \in \mathbb{C}^{2n},\ \displaystyle\sum_{g\in W} \langle \mathbf{w},\ g\mathbf{w}'\rangle\ P(\mathbf{w}+g\mathbf{w}') = 0\} & \rightarrow & \mathcal{L}_j \\ P = \displaystyle\sum_{m_j\in M_j(\mathbf{w})} \beta_{m_j} m_j & \mapsto & \left(\varphi_P\ :\ R_n(\overline{m_j}) \mapsto \frac{\beta_{m_j}}{\alpha_{m_j}}\right). \end{array} \tag{9}$$

$\triangleright$ For $f \in \mathcal{L}_j$, we have

$$\varphi_{\pi_f}\left(R_n(\overline{m_j})\right) = \frac{\alpha_{m_j} f\left(R_n(\overline{m_j})\right)}{\alpha_{m_j}} = f\left(R_n(\overline{m_j})\right),$$

thus $\varphi_{\pi_f} = f$.

$\triangleright$ For $P = \displaystyle\sum_{m_j\in M_j(\mathbf{w})} \beta_{m_j} m_j \in \mathbb{C}[\mathbf{w}]^W(j)$, we have $P = \displaystyle\sum_{m_j\in M_j(\mathbf{z})} \beta_{\widetilde{m_j}} \widetilde{m_j}$, so if $m_j \in M_j(\mathbf{z})$, then $\varphi_P(R_n(m_j)) = \frac{\beta_{\widetilde{m_j}}}{\alpha_{\widetilde{m_j}}}$.
Consequently,

$$\pi_{\varphi_P} = \sum_{m_j\in M_j(\mathbf{z})} \alpha_{\widetilde{m_j}} \varphi_P(R_n(m_j)) \widetilde{m_j} = \sum_{m_j\in M_j(\mathbf{z})} \alpha_{\widetilde{m_j}} \frac{\beta_{\widetilde{m_j}}}{\alpha_{\widetilde{m_j}}} \widetilde{m_j} = P.$$

So π is bijective and its inverse is φ.

$\triangleright$ All we have to do is to show that φ_P vanishes on $\{S^W,\ S^W\} \cap S^W(j)$.
Let $P \in \mathbb{C}[\mathbf{w}]^W(j)$ be a solution of equation (1). Then as, $\pi_{\varphi_P} = P$, we have for $k + l = j$,

$$0 = \sum_{g\in W} \langle \mathbf{w},\ g\mathbf{w}'\rangle P\left(\mathbf{w} + g\mathbf{w}'\right) = \varphi_P\left(\{L^k_{\mathbf{w}},\ L^l_{\mathbf{w}'}\}\right)$$

But

$$\{L^k_{\mathbf{w}},\ L^l_{\mathbf{w}'}\} = \sum_{m_k\in M_k(\mathbf{w})} \sum_{\mu_l\in M_l(\mathbf{w}')} \alpha_{m_k}\alpha_{\mu_l} m_k \mu_l \left\{R_n(\overline{m_k}),\ R_n(\overline{\mu_l})\right\},$$

so that

$$\sum_{m_k\in M_k(\mathbf{w})} \sum_{\mu_l\in M_l(\mathbf{w}')} \alpha_{m_k}\alpha_{\mu_l} m_k \mu_l\ \varphi_P\left(\{R_n(\overline{m_k}),\ R_n(\overline{\mu_l})\}\right) = 0.$$

This last equality is equivalent to

$$\forall\ k + l = j,\ \ \varphi_P\left(\{R_n(\overline{m_k}),\ R_n(\overline{\mu_l})\}\right) = 0,$$

which shows that φ_P vanishes on $\{S^W,\ S^W\} \cap S^W(j)$.■

The following corollary enables us to make the equation of Berest-Etingof-Ginzburg more explicit.

Corollary 10
We introduce $2n$ indeterminates, denoted by $z_1, \ldots,\ z_n,\ t_1, \ldots,\ t_n$, and we extend the Reynolds operator in a map from $\mathbb{C}[\mathbf{x},\ \mathbf{y},\ \mathbf{z},\ \mathbf{t}]$ to itself which is $\mathbb{C}[\mathbf{z},\ \mathbf{t}]-$linear. Then the vector space $\mathcal{L}_j$ is isomorphic to the vector space of polynomials $P \in S^W(j)$ satisfying the following equation :

$$R_n\Big(\big(\sum_{i=1}^{n} z_i y_i - t_i x_i\big)\ P(z_1 + x_1,\ \ldots,\ z_n + x_n,\ t_1 + y_1,\ \ldots,\ t_n + y_n)\Big) = 0 \tag{10}$$

i.e.

$$\boxed{E_n(P) := R_n\Big((\mathbf{z}\cdot\mathbf{y} - \mathbf{t}\cdot\mathbf{x})\ P(\mathbf{x}+\mathbf{z},\ \mathbf{y}+\mathbf{t})\Big) = 0} \tag{11}$$

<u>Proof :</u>
We have $\langle \mathbf{w},\ \mathbf{w}'\rangle = \mathbf{w}\cdot(J\mathbf{w}') = \sum_{i=1}^{n}(w_i w'_{n+i} - w_{n+i} w'_i)$. Then equation (10) is equivalent to

$$\begin{array}{rl} \forall\ \mathbf{w},\ \mathbf{w}' \in \mathbb{C}, & \sum_{g\in W} \sum_{i=1}^{n} (w_i \sum_{j=1}^{n} g_{ij} w'_{n+j} - w_{n+i} \sum_{j=1}^{n} g_{ij} w'_j) \\ & P\Big(w_1 + \sum_{j=1}^{n} g_{1j} w'_j,\ \ldots,\ w_n + \sum_{j=1}^{n} g_{nj} w'_j, \\ & w_{n+1} + \sum_{j=1}^{n} g_{1j} w'_{n+j},\ \ldots,\ w_{2n} + \sum_{j=1}^{n} g_{nj} w'_{n+j}\Big) = 0 \end{array} \tag{12}$$

This means that the polynomial

$$\sum_{g\in W}\sum_{i=1}^{n}(z_i\sum_{j=1}^{n} g_{ij}y_j - t_i\sum_{j=1}^{n} g_{ij}x_j)\; P\Big(z_1+\sum_{j=1}^{n} g_{1j}x_j,\;\ldots,\; z_n+\sum_{j=1}^{n} g_{nj}x_j,\; t_1+\sum_{j=1}^{n} g_{1j}y_j,\;\ldots,\; t_n+\sum_{j=1}^{n} g_{nj}y_j\Big) \quad (13)$$

is zero.
This is equivalent to

$$\sum_{i=1}^{n}(z_i\, g\cdot y_i - t_i\, g\cdot x_i)\;\sum_{g\in W} g\cdot\Big(P(z_1+x_1,\;\ldots,\; z_n+x_n,\; t_1+y_1,\;\ldots,\; t_n+y_n)\Big)=0, \quad (14)$$

that is to say

$$R_n\Big((\sum_{i=1}^{n} z_iy_i - t_ix_i)\; P(z_1+x_1,\;\ldots,\; z_n+x_n,\; t_1+y_1,\;\ldots,\; t_n+y_n)\Big)=0, \quad (15)$$

where R_n is the Reynolds operator extended in a $\mathbb{C}[\mathbf{z},\ \mathbf{t}]-$linear map.■

Remark 11
• *Case of B_n : for a monomial $M\in\mathbb{C}[\mathbf{x},\ \mathbf{y}]$,*
▷ *either there exists a sign change which sends M to its opposite, and then $R_n(M)=0$*
▷ *or M is invariant under every sign change and then $R_n(M)=\sum_{\sigma\in\mathfrak{S}_n}\sigma\cdot M$.*
If $Q=R_n(P)$ with $P\in\mathbb{C}[\mathbf{x},\ \mathbf{y}]$, then we may always assume that each monomial of P, in particular P itself, is invariant under the sign changes.
• *Case of D_n : we have the same result, by considering this time the sign changes of an even number of variables.*

The aim of Proposition 12 and its corollary is to reduce drastically the space in which we search the solutions of equation (11) : indeed, instead of searching the solutions in S^W, we may limit ourselves to the space of the elements which are annihilated by the action of $\mathfrak{sl}_2$.

Proposition 12
Let $P\in\mathbb{C}[\mathbf{x},\mathbf{y}]^W$. We consider the element $E_n(P)$ defined by the formula (11) as a polynomial in the indeterminates $\mathbf{z}$, $\mathbf{t}$ and with coefficients in $\mathbb{C}[\mathbf{x},\mathbf{y}]$.
Then the coefficient of z_1t_1 in $E_n(P)$ is $\frac{-1}{n}\{H,\,P\}$, that of t_1^2 is $\frac{-1}{n}\{E,\,P\}$ and that of z_1^2 is $\frac{1}{n}\{F,\,P\}$.

Proof :
We carry out the proof for B_n. The method is the same for D_n.
• We denote by $c_{z_1t_1}(P)$ the coefficient of z_1t_1 in $E_n(P)$. Since the maps $P\mapsto c_{z_1t_1}(P)$ and $P\mapsto\{H,\,P\}$ are linear, all we have to do is to prove the property for P of the form $P=R_n(M)$, where $M=\mathbf{x}^{\mathbf{i}}\mathbf{y}^{\mathbf{j}}$ is a monomial which we may assume invariant under the sign changes thanks to remark 11.
Then the formula (11) may be written

$$\begin{aligned}
|W|\,E_n(M) &= |W|\,R_n\left((\mathbf{z}\cdot\mathbf{y}-\mathbf{t}\cdot\mathbf{x})(\mathbf{x}+\mathbf{z})^{\mathbf{i}}(\mathbf{y}+\mathbf{t})^{\mathbf{j}}\right)\\
&= \sum_{c\in(\pm1)^n}\sum_{\sigma\in\mathfrak{S}_n} c\cdot\left[(z_1y_{\sigma^{-1}(1)}+\cdots+z_ny_{\sigma^{-1}(n)})\prod_{k=1}^{n}(z_k+x_{\sigma^{-1}(k)})^{i_k}(t_k+y_{\sigma^{-1}(k)})^{j_k}\right]\\
&\quad-\sum_{c\in(\pm1)^n}\sum_{\sigma\in\mathfrak{S}_n} c\cdot\left[(t_1x_{\sigma^{-1}(1)}+\cdots+t_nx_{\sigma^{-1}(n)})\prod_{k=1}^{n}(z_k+x_{\sigma^{-1}(k)})^{i_k}(t_k+y_{\sigma^{-1}(k)})^{j_k}\right]
\end{aligned}$$

• So the coefficient of z_1t_1 is given by

$$\begin{aligned}
|W|\,c_{z_1t_1}(M) &= \sum_{c\in(\pm1)^n}\sum_{\sigma\in\mathfrak{S}_n} c\cdot\Big[y_{\sigma^{-1}(1)}\left(\prod_{k=1}^{n}x^{i_k}_{\sigma^{-1}(k)}\right)j_1\,y^{j_1-1}_{\sigma^{-1}(1)}\left(\prod_{k=2}^{n}y^{j_k}_{\sigma^{-1}(k)}\right)\\
&\qquad -x_{\sigma^{-1}(1)}\,i_1\,x^{i_1-1}_{\sigma^{-1}(1)}\left(\prod_{k=2}^{n}x^{i_k}_{\sigma^{-1}(k)}\right)\left(\prod_{k=1}^{n}y^{j_k}_{\sigma^{-1}(k)}\right)\Big]\\
&= |(\pm1)^n|\sum_{\sigma\in\mathfrak{S}_n}(j_1-i_1)\left(\prod_{k=1}^{n}x^{i_k}_{\sigma^{-1}(k)}y^{j_k}_{\sigma^{-1}(k)}\right)\\
&= |W|\,(j_1-i_1)\,R_n(M).
\end{aligned}$$

Since $P = \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n}\prod_{k=1}^n x_k^{i_{\sigma(k)}} y_k^{j_{\sigma(k)}}$, we deduce that

$$\begin{aligned}
n!\, c_{z_1 t_1}(P) &= \sum_{\sigma\in\mathfrak{S}_n} c_{z_1 t_1}\left(\prod_{k=1}^n x_k^{i_{\sigma(k)}} y_k^{j_{\sigma(k)}}\right) = \sum_{\sigma\in\mathfrak{S}_n} (j_{\sigma(1)} - i_{\sigma(1)})\, R_n\left(\prod_{k=1}^n x_k^{i_{\sigma(k)}} y_k^{j_{\sigma(k)}}\right)\\
&= \sum_{\sigma\in\mathfrak{S}_n} (j_{\sigma(1)} - i_{\sigma(1)})\, R_n\,(M) = (n-1)!\sum_{k=1}^n (j_k - i_k)\, R_n\,(M)\\
&= (n-1)!\,(\deg_y(M) - \deg_x(M))\, R_n\,(M) = -(n-1)!\,\{H,\, P\}.
\end{aligned}$$

- We proceed as for $z_1 t_1$, by denoting by $c_{t_1^2}(P)$ the coefficient of t_1^2 in $E_n(P)$. Then we have

$$\begin{aligned}
|W|\, c_{t_1^2}(M) &= -\sum_{c\in(\pm 1)^n}\sum_{\sigma\in\mathfrak{S}_n} c\cdot\left[x_{\sigma^{-1}(1)}\left(\prod_{k=1}^n x_{\sigma^{-1}(k)}^{i_k}\right) j_1\, y_{\sigma^{-1}(1)}^{j_1-1}\left(\prod_{k=2}^n y_{\sigma^{-1}(k)}^{j_k}\right)\right]\\
&= -|(\pm 1)^n|\, j_1 \sum_{\sigma\in\mathfrak{S}_n} x_{\sigma^{-1}(1)}^{i_1+1} y_{\sigma^{-1}(1)}^{j_1-1}\left(\prod_{k=2}^n x_{\sigma^{-1}(k)}^{i_k} y_{\sigma^{-1}(k)}^{j_k}\right)\\
&= -|W|\, j_1\; R_n\left(x_1^{i_1+1} y_1^{j_1-1}\left(\prod_{k=2}^n x_k^{i_k} y_k^{j_k}\right)\right).
\end{aligned}$$

Thus

$$\begin{aligned}
n!\; c_{t_1^2}(P) &= \sum_{\sigma\in\mathfrak{S}_n} c_{t_1^2}\left(\prod_{k=1}^n x_k^{i_{\sigma(k)}} y_k^{j_{\sigma(k)}}\right) = -\sum_{\sigma\in\mathfrak{S}_n} j_{\sigma(1)}\, R_n\left(x_1^{i_{\sigma(1)}+1} y_1^{j_{\sigma(1)}-1}\left(\prod_{k=2}^n x_k^{i_{\sigma(k)}} y_k^{j_{\sigma(k)}}\right)\right)\\
&= -\sum_{p=1}^n \sum_{\substack{\sigma\in\mathfrak{S}_n\\ \sigma(1)=p}} j_{\sigma(1)}\, R_n\left(x_1^{i_{\sigma(1)}+1} y_1^{j_{\sigma(1)}-1}\left(\prod_{k=2}^n x_k^{i_{\sigma(k)}} y_k^{j_{\sigma(k)}}\right)\right)\\
&= -(n-1)!\sum_{p=1}^n j_p\, R_n\left(x_1^{i_1}\dots x_p^{i_p+1}\dots x_n^{i_n} y_1^{j_1}\dots y_p^{j_p-1}\dots y_n^{j_n}\right).
\end{aligned}$$

But

$$\begin{aligned}
n!\;\{E,\, P\} &= \sum_{\sigma\in\mathfrak{S}_n}\{E,\, x_1^{i_{\sigma(1)}}\dots x_n^{i_{\sigma(n)}} y_1^{j_{\sigma(1)}}\dots y_n^{j_{\sigma(n)}}\}\\
&= \sum_{\sigma\in\mathfrak{S}_n}\sum_{p=1}^n j_{\sigma(p)} x_1^{i_{\sigma(1)}}\dots x_p^{i_{\sigma(p)}+1}\dots x_n^{i_{\sigma(n)}} y_1^{j_{\sigma(1)}}\dots y_p^{j_{\sigma(p)}-1}\dots y_n^{j_{\sigma(n)}}\\
&= \sum_{p=1}^n\sum_{q=1}^n\sum_{\substack{\sigma\in\mathfrak{S}_n\\ \sigma(p)=q}} j_{\sigma(p)} x_1^{i_{\sigma(1)}}\dots x_p^{i_{\sigma(p)}+1}\dots x_n^{i_{\sigma(n)}} y_1^{j_{\sigma(1)}}\dots y_p^{j_{\sigma(p)}-1}\dots y_n^{j_{\sigma(n)}}\\
&= \sum_{q=1}^n j_q \sum_{p=1}^n\sum_{\substack{\sigma\in\mathfrak{S}_n\\ \sigma(p)=q}} x_1^{i_{\sigma(1)}}\dots x_p^{i_{\sigma(p)}+1}\dots x_n^{i_{\sigma(n)}} y_1^{j_{\sigma(1)}}\dots y_p^{j_{\sigma(p)}-1}\dots y_n^{j_{\sigma(n)}}\\
&= n!\sum_{q=1}^n j_q\, R_n\left(x_1^{i_1}\dots x_q^{i_q+1}\dots x_n^{i_n} y_1^{j_1}\dots y_q^{j_q-1}\dots y_n^{j_n}\right).
\end{aligned}$$

So $c_{t_1^2}(P) = \frac{-1}{n}\{E,\, P\}$. Similarly we show that $c_{z_1^2}(P) = \frac{1}{n}\{F,\, P\}$.■

Corollary 13

- *Let $P \in \mathbb{C}[\mathbf{x},\, \mathbf{y}]^W$. If P satisfies equation (11), then P is annihilated by $\mathfrak{sl}_2$, i.e. $P \in S^W_{\mathfrak{sl}_2}$.*
- *Therefore the vector space $\mathcal{L}_j$ is isomorphic to the vector space of the polynomials $P \in S^W_{\mathfrak{sl}_2}(j)$ satisfying equation (11).*

Thus the determination of $\frac{S^W}{\{S^W,\, S^W\}}\left(= \frac{S^W_{\mathfrak{sl}_2}}{\{S^W,\, S^W\}\cap S^W_{\mathfrak{sl}_2}}\right)$ is equivalent to the resolution, in $S^W_{\mathfrak{sl}_2}$, of equation (11).

Proof :

Let $P \in \mathbb{C}[\mathbf{x},\, \mathbf{y}]^W$ satisfying equation (11). Then all the coefficients of the polynomial $E_n(P) \in (\mathbb{C}[\mathbf{x},\, \mathbf{y}])[\mathbf{z},\, \mathbf{t}]$ are zero. In particular, according to Proposition 12, we have $\{H,\, P\} = \{E,\, P\} = \{F,\, P\} = 0$. Hence $P \in S^W_{\mathfrak{sl}_2}$.
The second point results from Corollary 10 and from the first point.■

We end this section by defining two variants of the equation of Berest-Etingof-Ginzburg : these are technical tools which enable us to eliminate some variables and thus to solve equation (11) more easily.

Definition 14
We define the (intermediate) map

$$\begin{array}{rcl} s^n_{int} \ : \ \mathbb{C}[\mathbf{x},\ \mathbf{y},\ \mathbf{z},\ \mathbf{t}] & \rightarrow & \mathbb{C}[\mathbf{x},\ \mathbf{y},\ \mathbf{z},\ \mathbf{t}] \\ P & \mapsto & P(\mathbf{0}\ \ \mathbf{y}\ \ \mathbf{z}\ \ t_1,\ \mathbf{0}), \end{array}$$

and we set

$$E^n_{int}(P) := s^n_{int}(E_n(P)). \tag{16}$$

Similarly, we define the map

$$\begin{array}{rcl} s_n \ : \ \mathbb{C}[\mathbf{x},\ \mathbf{y},\ \mathbf{z},\ \mathbf{t}] & \rightarrow & \mathbb{C}[\mathbf{x},\ \mathbf{y},\ \mathbf{z},\ \mathbf{t}] \\ P & \mapsto & P(\mathbf{0}\ \ y_1,\ \mathbf{0}\ \ \mathbf{z}\ \ t_1,\ \mathbf{0}), \end{array}$$

and we set

$$E'_n(P) := s_n(E_n(P)). \tag{17}$$

This last equation is equation (11) after the substitution

$$x_1 = \cdots = x_n = y_2 = \cdots = y_n = t_2 = \cdots = t_n = 0.$$

Remark 15
If P satisfies equation (11), it satisfies obviously equation (17).

In the case where n is an odd integer, the vectors of highest weight 0 of even degree are the same for B_n and D_n, and equations (17) are identical for both types. Moreover, the link between equations (11) for B_n and D_n enables us to prove the inequality dim $HP_0(D_n) \leq$ dim $HP_0(B_n)$. It is the purpose of the two following propositions.

Proposition 16
By abuse of notation, we denote by $S^{B_n}(2p)$ (resp. $S^{D_n}(2p)$) the set of invariant elements of degree $2p$ in type B_n (resp. D_n). Then we have $S^{B_{2n+1}}(2p) = S^{D_{2n+1}}(2p)$, $S^{B_{2n+1}}_{\mathfrak{sl}_2}(p) = S^{D_{2n+1}}_{\mathfrak{sl}_2}(p)$, and equations (17) in $S^{B_{2n+1}}_{\mathfrak{sl}_2} = S^{D_{2n+1}}_{\mathfrak{sl}_2}$ associated to both types are the same.
This result is false for the even indices : counter-example : dim $S^{D_4}_{\mathfrak{sl}_2}(6) = 1$ *whereas* $S^{B_4}_{\mathfrak{sl}_2}(6) = \{0\}$.

Proof :
• We set

$$\Phi_{2n+1}(P) = \sum_{\sigma \in \mathfrak{S}_{2n+1}} \sigma \cdot P, \qquad \Psi^B_{2n+1}(P) = \sum_{g \in (\pm 1)^{2n+1}} g \cdot P, \qquad \Psi^D_{2n+1}(P) = \sum_{g \in (\pm 1)^{2n}} g \cdot P,$$

so that

$$R^B_{2n+1}(P) = \frac{1}{|B_{2n+1}|}\Phi_{2n+1} \circ \Psi^B_{2n+1}, \quad \text{and} \quad R^D_{2n+1}(P) = \frac{1}{|D_{2n+1}|}\Phi_{2n+1} \circ \Psi^D_{2n+1}.$$

We obviously have $\Psi^B_{2n+1}(S(2p)) \subset \Psi^D_{2n+1}(S(2p))$.
Conversely, since $\Psi^D_{2n+1}(S(2p))$ is spanned by the elements of the form $\Psi^D_{2n+1}(\mathbf{m})$ with $\mathbf{m} \in S(2p)$ monomial, all we have to do is to show that $\Psi^D_{2n+1}(\mathbf{m})$ belongs to $\Psi^B_{2n+1}(S(2p))$, i.e. $\Psi^D_{2n+1}(\mathbf{m})$ is invariant under the sign changes. Now $\mathbf{m} = x_1^{i_1} \ldots x_{2n+1}^{i_{2n+1}} y_1^{j_1} \ldots y_{2n+1}^{j_{2n+1}}$ with $\sum_{k=1}^{2n+1}(i_k + j_k) = 2p$, therefore at least one of the $i_k + j_k$ is even. Let's denote by l the corresponding index.
So, for every $k \neq l$, we have $s_k(\mathbf{m}) = (-1)^{i_k+j_k}\mathbf{m} = s_{k,l}(\mathbf{m})$ and $s_l(\mathbf{m}) = \mathbf{m}$. But

$$\Psi^D_{2n+1}(\mathbf{m}) = \underbrace{\left(\sum_{q_1=0,1 \ldots q_{2n+1}=0,1} (-1)^{q_1[(i_1+j_1)+(i_2+j_2)]+q_2[(i_2+j_2)+(i_3+j_3)]+\cdots+q_{2n}[(i_{2n}+j_{2n})+(i_{2n+1}+j_{2n+1})]} \right)}_{a_{\mathbf{m}}} \mathbf{m},$$

therefore $s_k\left(\Psi^D_{2n+1}(\mathbf{m})\right) = \left\{ \begin{array}{ll} a_{\mathbf{m}} s_{k,l}(\mathbf{m}) = s_{k,l}(a_{\mathbf{m}}\mathbf{m}) = s_{k,l}\left(\Psi^D_{2n+1}(\mathbf{m})\right) & \text{si } k \neq l \\ a_{\mathbf{m}}\mathbf{m} & \text{si } k = l \end{array} \right\} = \Psi^D_{2n+1}(\mathbf{m})$.
• So we have $S^{D_n}(2p) = \Phi_{2n+1}\left(\Psi^D_{2n+1}(S(2p))\right) = \Phi_{2n+1}\left(\Psi^B_{2n+1}(S(2p))\right) = S^{B_n}(2p)$.
Hence $S^{B_n}_{\mathfrak{sl}_2}(2p) = S^{D_n}_{\mathfrak{sl}_2}(2p)$. Besides, according to Proposition 5, $S^{B_n}_{\mathfrak{sl}_2}(2p+1) = S^{D_n}_{\mathfrak{sl}_2}(2p+1) = \{0\}$.
• For $P \in S^{B_n}_{\mathfrak{sl}_2}$, equation (17) may be written

$$\sum_{i=1}^{2n+1} z_i \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \sigma(1)=i}} \left[y_1\, P(z_1, \ldots,\ z_{2n+1},\ t_1,\ 0, \ldots,\ 0,\ \underbrace{y_1}_{i},\ 0, \ldots,\ 0) - y_1\, P(z_1, \ldots,\ z_{2n+1},\ t_1,\ 0, \ldots,\ 0,\ \underbrace{-y_1}_{i},\ 0, \ldots,\ 0) \right] = 0.$$

It is equation (17) for $P \in S^{D_n}_{\mathfrak{sl}_2}$.■

Proposition 17
Let P be invariant by sign changes. If P is solution of equation (11) for D_n, then P is solution of equation (11) for B_n.
In particular, we have $\dim\, HP_0(D_{2n+1}) \leq \dim\, HP_0(B_{2n+1})$.

Proof :
Let SB_n (resp. SD_n) be the group of sign changes of B_n (resp. D_n). We may write $SB_n = SD_n \sqcup SD_n \cdot s_1$.
Let P be invariant by sign changes. So we have $P = R_n^B(P) = R_n^D(P)$, and equation (11) for B_n (resp. D_n) may be written $E_n^B(P) = R_n^B(Q)$ (resp. $E_n^D(P) = R_n^D(Q)$), with $Q = (\mathbf{z}\cdot\mathbf{y} - \mathbf{t}\cdot\mathbf{x})\, P(\mathbf{x} + \mathbf{z}\,\mathbf{y} + \mathbf{t})$.
If P is solution of equation (11) for D_n, then we have :

$$\begin{aligned} R_n^B(Q) &= \sum_{h\in SB_n}\sum_{\sigma\in\mathfrak{S}_n}(\sigma h)\cdot Q = \sum_{h\in SB_n} h\cdot\left(\sum_{\sigma\in\mathfrak{S}_n}\sigma\cdot Q\right) \\ &= \sum_{g\in SD_n} g\cdot\left(\sum_{\sigma\in\mathfrak{S}_n}\sigma\cdot Q\right) + \sum_{g\in SD_n}(gs_1)\cdot\left(\sum_{\sigma\in\mathfrak{S}_n}\sigma\cdot Q\right) \\ &= \sum_{g\in SD_n} g\cdot\left(\sum_{\sigma\in\mathfrak{S}_n}\sigma\cdot Q\right) + s_1\cdot\left[\sum_{g\in SD_n} g\cdot\left(\sum_{\sigma\in\mathfrak{S}_n}\sigma\cdot Q\right)\right] \\ &= R_n^D(Q) + s_1\cdot R_n^D(Q) = 0. \end{aligned}$$

So, P is solution of equation (11) for B_n.
We deduce the claimed inequality, knowing that, according to Proposition 16, $S_{\mathfrak{sl}_2}^{B_{2n+1}} = S_{\mathfrak{sl}_2}^{D_{2n+1}}$. ■

2.5 Construction of graphs attached to the invariant polynomials

Let us recall the equality of Proposition 8 : $S_{\mathfrak{sl}_2}^W = R_n\left(\mathbb{C}\langle X_{i,j}\rangle\right)$. Moreover, according to Corollary 13, the computation of $HP_0(S^W)$ can be reduces to solving Equation (11) in the space $S_{\mathfrak{sl}_2}^W$.
In order to have shorter and more visual notations, we represent the polynomials of this space by graphs, by the method explained in definition 21.
But before, let us quote, for the particular case that we are interested in, the fundamental result established by J. Alev, M. A. Farinati, T. Lambre and A. L. Solotar in [AFLS00] :

Theorem 18 *(Alev-Farinati-Lambre-Solotar)*
For $k = 0\ldots 2n$, the dimension of $HH_k(A_n(\mathbb{C})^W)$ is the number of conjugacy classes of W admitting the eigenvalue 1 with the multiplicity k.

By specializing to the cases of B_n and D_n, we obtain :

Corollary 19 *(Alev-Farinati-Lambre-Solotar)*
• *For type B_n, the dimension of $HH_0(A_n(\mathbb{C})^W)$ is the number of partitions $\pi(n)$ of the integer n.*
• *For type D_n, the dimension of $HH_0(A_n(\mathbb{C})^W)$ is the number of partitions $\widetilde{\pi}(n)$ of the integer n having an even number of parts.*

The conjecture of J. Alev may be set forth as follows :

Conjecture 20 *(Alev)*
• *For the type B_n, the dimension of $HP_0(S^W)$ equals the number of partitions $\pi(n)$ of the integer n.*
• *For the type D_n, the dimension of $HP_0(S^W)$ equals the number of partitions $\widetilde{\pi}(n)$ of the integer n having an even number of parts.*

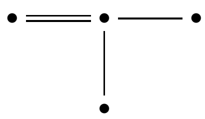
Now, let us show how to construct $\pi(n)$ solutions of equation (11) for the case of B_n.

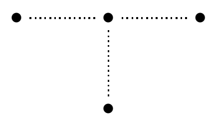
Definition 21
For $i \neq j$, we note $X_{i,j} = x_i y_j - y_i x_j$.
To each element of the form $M := \prod_{i=1}^{n-1}\prod_{j=i+1}^{n} X_{i,j}^{2a_{i,j}}$, we associate the (non-oriented) graph $\widetilde{\Gamma_M}$ such that
▷ *the set of vertices of $\widetilde{\Gamma_M}$ is the set of indices* $\{k \in [\![1,\, n]\!] \;/\; \exists\, i \in [\![1,\, n]\!] \;/\; a_{i,k} \neq 0 \text{ or } a_{k,i} \neq 0\}$,
▷ *two vertices i, j of $\widetilde{\Gamma_M}$ are connected by the edge* $i \overset{a_{i,j}}{\longrule} j$ *if* $a_{i,j} \neq 0$.

• If $\sigma \in \mathfrak{S}_n$, then the graph $\widetilde{\Gamma_{\sigma\cdot M}}$ is obtained by permuting the vertices of $\widetilde{\Gamma_M}$.
So, by replacing each vertex by the symbol •, we obtain a graph Γ_M such that the map $M \mapsto \Gamma_M$ is constant on every orbit under the action of B_n (resp. D_n). So we may associate this graph to the element $R_n\left(\prod_{i=1}^{n-1}\prod_{j=i+1}^{n} X_{i,j}^{2a_{i,j}}\right)$.

To a linear combination $\sum_{k=1}^{p} \alpha_k M_k$, we associate the graph $\sum_{k=1}^{p} \alpha_k \, \Gamma_{M_k}$.

• We may extend this definition to elements of the form $M := \prod_{i=1}^{n-1} \prod_{j=i+1}^{n} X_{i,j}^{b_{i,j}}$ by denoting an edge by $\bullet \overset{\frac{b_{i,j}}{2}}{\text{———}} \bullet$ if $b_{i,j}$ is even and by $\bullet \overset{\frac{b_{i,j}+1}{2}}{\cdots\cdots} \bullet$ if $b_{i,j}$ is odd. Be careful ! We have for example $\bullet \cdots\cdots \bullet = 0$.

Example 22
The polynomial $R_4(X_{1,2}^4 X_{1,3}^2 X_{1,4}^2)$ is represented by the graph  .

• If a graph contains only even edges (i.e. of the form $\bullet \overset{a_{i,j}}{\text{———}} \bullet$), then it is represented in B_n and in D_n by the same element.
This result is not valid in the case of graphs which contain odd edges : for example, the element 
is zero in B_4, but different from zero in D_4.

Remark 23
The graphs corresponding to polynomials obtained by operating the Reynolds operator for different indices on the same elements of the algebra generated by the $X_{i,j}$'s are the same.
For example, the elements $R_3(X_{1,2}^2)$ and $R_{44}(X_{1,2}^2)$ are represented in this way by the same graph $\bullet \text{———} \bullet$.
Propositions 25 and 26 show that this has no effect on the study of equation (11) for B_n.

Proposition 24
For every $n \in \mathbb{N}^*$, the number of linear graphs without loops and without isolated vertices is equal to the number of partitions of n. (A multiple edge is viewed as a loop).

Proof : immediate. To each partition p of the form $n = 1p_1 + 2p_2 + 3p_3 + \cdots + np_n$, we associate the graph having p_j linear connected components with j vertices.

Proposition 25
• *Let $P \in \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$. If $R_n(P) \neq 0$ then $R_{n+1}(P) \neq 0$.*
• *Let $P_1, \ldots, P_m \in \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$. If $R_n(P_1), \ldots, R_n(P_m)$ are linearly independent, then $R_{n+1}(P_1), \ldots, R_{n+1}(P_m)$ are linearly independent.*

Proof :
• We carry out the proof for B_n. We proceed likewise for D_n.
Let $P \in \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ such that $R_n(P) \neq 0$. According to remark 11, we may assume that the terms of P are invariant under sign changes.
We consider the set $\mathcal{T}_n$ of the terms of $R_n(P)$ that we partition into orbits under the action of $\mathfrak{S}_n$: so we have the equality $\mathcal{T}_n = \coprod_{j=1}^{r} \mathcal{O}_j$. Consequently, $R_n(P)$ may be written

$$R_n(P) = \sum_{j=1}^{r} \alpha_j R_n(M_j),$$

where $M_j \in \mathcal{O}_j$.
Let $c_{n+1} := (1, \ldots, n+1) \in \mathfrak{S}_{n+1}$, and s_{n+1} the $(n+1)-$th sign change, so that $B_{n+1} = \langle s_{n+1}, c_{n+1} \rangle \cdot B_n$. Again by the invariance under sign changes, we deduce

$$R_{n+1}(P) = \frac{1}{n+1} \sum_{j=1}^{r} \alpha_j \underbrace{\left(\sum_{k=0}^{n} c_{n+1}^{k} \cdot R_n(M_j) \right)}_{t_j}.$$

Now if $i \neq j$, then t_i and t_j belong to two distinct orbits under the action of $\mathfrak{S}_{n+1}$, Therefore the t_j's are linearly independent. So $R_{n+1}(P) \neq 0$.
• Let be $P_1, \ldots, P_m \in \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ such that $R_n(P_1), \ldots, R_n(P_m)$ are linearly independent. Let's consider a zero linear combination $\sum_{j=1}^{m} \lambda_j R_{n+1}(P_j) = 0$, i.e. $R_{n+1}\left(\sum_{j=1}^{m} \lambda_j P_j\right) = 0$. Then, according to the

first point, we have $\sum_{j=1}^m \lambda_j R_n(P_j) = R_n\left(\sum_{j=1}^m \lambda_j P_j\right) = 0$, so by hypothesis, $\forall\, j \in [\![1,\ m]\!],\ \lambda_j = 0.$■

The following proposition shows the fact that, for a graph, being solution of equation (11) for B_n is independent of n, provided that n is not smaller than the number of vertices of the graph! This proposition justifies the n-independent notation of graphs.

Proposition 26
Let $P \in \mathbb{C}[x_1, \dots,\ x_n,\ y_1, \dots,\ y_n]$.
• *Case of B_n : if $R_n(P)$ satisfies the equation $E_n\left(R_n(P)\right) = 0$, then $R_{n+1}(P) \in \mathbb{C}[x_1, \dots,\ x_{n+1},\ y_1, \dots,\ y_{n+1}]$ satisfies the equation $E_{n+1}\left(R_{n+1}(P)\right) = 0$.*
• *Case of D_n : if $R_n(P)$ satisfies the equation $E_n\left(R_n(P)\right) = 0$, then $R_{n+2}(P) \in \mathbb{C}[x_1, \dots,\ x_{n+2},\ y_1, \dots,\ y_{n+2}]$ satisfies the equation $E_{n+2}\left(R_{n+2}(P)\right) = 0$.*

Proof :
We carry out the proof for B_n ; we proceed likewise for D_n.
According to remark 11, we may assume that P is invariant by sign changes.
For every $n \geq 2$, we note $Q_n := R_n(P)$. Then

$$E_{n+1}(Q_{n+1}) = R_{n+1}\left(\left(\sum_{k=1}^{n+1} z_k y_k - t_k x_k\right) Q_{n+1}(x_1+z_1, \dots, x_{n+1}+z_{n+1}, y_1+t_1, \dots, y_{n+1}+t_{n+1})\right).$$

Let $c_{n+1} := (1,\ \dots,\ n+1) \in \mathfrak{S}_{n+1}$, and s_{n+1} the $(n+1)-$th sign change, so that $B_{n+1} = \langle s_{n+1},\ c_{n+1}\rangle \cdot B_n$. Then we may write

$$Q_{n+1} = R_{n+1}(P) = \frac{1}{2(n+1)} \sum_{\substack{i=0,1\\ j=1\dots n+1}} (s_{n+1}^i c_{n+1}^j) \cdot Q_n.$$

Now the polynomial $(s_{n+1}^i c_{n+1}^j) \cdot Q_n$ contains only the indices $1, \dots,\ j-1,\ j+1, \dots,\ n+1$, therefore $(z_j y_j - t_j x_j)(s_{n+1}^i c_{n+1}^j) \cdot Q_n$ is in the kernel of R_{n+1}. So,

$$E_{n+1}(Q_{n+1}) = \frac{1}{2(n+1)} \sum_{\substack{i=0,1\\ j=1\dots n+1}} R_{n+1}\left[\left(\sum_{\substack{k=1\\ k\neq j}}^{n+1} z_k y_k - t_k x_k\right)\left((s_{n+1}^i c_{n+1}^j)\cdot Q_n\right)(\mathbf{x}+\mathbf{z},\ \mathbf{y}+\mathbf{t})\right],$$

i.e.

$$\begin{aligned} E_{n+1}(Q_{n+1}) &= \frac{1}{2(n+1)} \sum_{\substack{i=0,1\\ j=1\dots n+1}} R_{n+1}\left[(\widetilde{s_{n+1}}^i \widetilde{c_{n+1}}^j)\cdot\left(\left(\sum_{k=1}^{n} z_k y_k - t_k x_k\right) Q_n(\mathbf{x}+\mathbf{z},\ \mathbf{y}+\mathbf{t})\right)\right] \\ &= \frac{1}{n+1}\sum_{j=1}^{n+1} R_{n+1}\left[\widetilde{c_{n+1}}^j\cdot\left(\left(\sum_{k=1}^{n} z_k y_k - t_k x_k\right) Q_n(\mathbf{x}+\mathbf{z},\ \mathbf{y}+\mathbf{t})\right)\right], \end{aligned}$$

where the $\widetilde{s_{n+1}}$ and $\widetilde{c_{n+1}}$ act on the $\mathbf{x},\ \mathbf{y},\ \mathbf{z},\ \mathbf{t}$ as the s_{n+1} and c_{n+1} act on the $\mathbf{x},\ \mathbf{y}$. (In the 2−nd equality, we have used the invariance by sign changes of P).
By indexing the variables by $[\![1,\ n]\!]$ instead of $[\![1,\ n+1]\!]\backslash\{j\}$, we see that each of the terms of this sum is by hypothesis in the kernel of R_n, thus in the kernel of R_{n+1}. So, $E_{n+1}(Q_{n+1}) = 0.$■

Corollary 27
The sequence $(\dim\ HP_0(B_n))_{n\geq 2}$ *is increasing.*
The sequences $(\dim\ HP_0(D_{2n}))_{n\geq 1}$ *and* $(\dim\ HP_0(D_{2n+1}))_{n\geq 1}$ *are increasing.*

Proposition 28 is fundamental for the construction of the solutions of the equation of Berest-Etingof-Ginzburg for B_n.

Proposition 28
If $R_i(P)$ and $R_j(Q)$ satisfy equation (11), and if their sets of indeterminates are disjoints*, then $R_{i+j}(PQ)$ satisfies equation (11).*
In terms of graphs, it means that if two disjoints *graphs satisfy equation (11), then their union also satisfies this equation.*
So it is sufficient that the connected components of a graph satisfy equation (11) in order that the graph itself satisfies it.

Proof :

• Let $R_i(P)$ and $R_j(Q)$ of which the sets of indeterminates are disjoints.
We denote by $E_P := \{x_k,\ k \in I_P\} \cup \{y_k,\ k \in I_P\}$ (resp. $E_Q := \{x_k,\ k \in I_Q\} \cup \{y_k,\ k \in I_Q\}$) the set of indeterminates of P (resp. Q), and we set $n := i + j$ so that we have $|E_P| = 2i$, $|E_Q| = 2j$, $I_P \sqcup I_Q = [\![1,\ n]\!]$ and $\mathbb{C}[E_P,\ E_Q] = \mathbb{C}[\mathbf{x},\ \mathbf{y}]$. The group B_i (resp. B_j) acts only on the indeterminates of E_P (resp. E_Q).
Let us calculate $R_n(PQ)$:

$$\begin{aligned}
|B_n| R_n(PQ) &= \frac{1}{|B_i||B_j|} \sum_{g\in B_i} \sum_{h\in B_j} \sum_{\sigma\in B_n} (gh\sigma)\cdot(PQ) \\
&= \frac{1}{|B_i||B_j|} \sum_{\sigma\in B_n} \sigma \cdot \left[\sum_{g\in B_i} \sum_{h\in B_j} \underbrace{(h\cdot(g\cdot P))}_{g\cdot P} \underbrace{(h\cdot(g\cdot Q))}_{h\cdot Q} \right] \\
&= \frac{1}{|B_i||B_j|} \sum_{\sigma\in B_n} \sigma \cdot \left[\left(\sum_{g\in B_i} g\cdot P \right) \left(\sum_{h\in B_j} h\cdot Q \right) \right] \\
&= \sum_{\sigma\in B_n} \sigma \cdot [R_i(P)R_j(Q)] \\
&= |B_n| R_n\left(R_i(P)R_j(Q)\right).
\end{aligned}$$

Hence $R_n(PQ) = R_n\left(R_i(P)R_j(Q)\right)$.

• If $R_i(P)$ and $R_j(Q)$ satisfy moreover equation (11), we have

$$\begin{aligned}
& (\mathbf{z}\cdot\mathbf{y} - \mathbf{t}\cdot\mathbf{x})\,[R_n(PQ)]\,(\mathbf{x}+\mathbf{z},\ \mathbf{y}+\mathbf{t}) \\
= \ & \frac{1}{|B_n|} \sum_{\sigma\in B_n} (\mathbf{z}\cdot\mathbf{y} - \mathbf{t}\cdot\mathbf{x})\,[\sigma\cdot(R_i(P)R_j(Q))]\,(\mathbf{x}+\mathbf{z},\ \mathbf{y}+\mathbf{t}) \\
= \ & \frac{1}{|B_n|} \sum_{\sigma\in B_n} \Bigg[\underbrace{\left(\sum_{k\in\sigma^{-1}(I_P)} z_k y_k - t_k x_k \right) [\sigma\cdot R_i(P)](\mathbf{x}+\mathbf{z},\ \mathbf{y}+\mathbf{t})}_{A_{P,\sigma}} \ \underbrace{[\sigma\cdot R_j(Q)](\mathbf{x}+\mathbf{z},\ \mathbf{y}+\mathbf{t})}_{A_{Q,\sigma}} \\
& + \underbrace{\left(\sum_{k\in\sigma^{-1}(I_Q)} z_k y_k - t_k x_k \right) [\sigma\cdot R_j(Q)](\mathbf{x}+\mathbf{z},\ \mathbf{y}+\mathbf{t})}_{B_{Q,\sigma}} \ \underbrace{[\sigma\cdot R_i(P)](\mathbf{x}+\mathbf{z},\ \mathbf{y}+\mathbf{t})}_{B_{P,\sigma}} \Bigg],
\end{aligned}$$

where the elements $A_{P,\sigma}$ and $B_{P,\sigma}$ (resp. $A_{Q,\sigma}$ and $B_{Q,\sigma}$) contain only the indices of $\sigma^{-1}(I_P)$ (resp. $\sigma^{-1}(I_Q)$).

• Let us show that $A_{P,\sigma}A_{Q,\sigma}$ is in the kernel of R_n :

$$\begin{aligned}
|B_n|\, R_n(A_{P,\sigma}A_{Q,\sigma}) &= \sum_{g\in B_n} g\cdot(A_{P,\sigma}A_{Q,\sigma}) \\
&= \frac{1}{|B_i|} \sum_{h\in B_i} \sum_{g\in hB_n} g\cdot(A_{P,\sigma}A_{Q,\sigma}) \\
&= \frac{1}{|B_i|} \sum_{g\in B_n} g\cdot \left(\sum_{h\in B_i} h\cdot(A_{P,\sigma}A_{Q,\sigma}) \right) \\
&= \frac{1}{|B_i|} \sum_{g\in B_n} g\cdot \left(\left(\sum_{h\in B_i} h\cdot A_{P,\sigma} \right) A_{Q,\sigma} \right) \\
&= \sum_{g\in B_n} g\cdot \left(\underbrace{R_i(A_{P,\sigma})}_{=0} A_{Q,\sigma} \right) = 0.
\end{aligned}$$

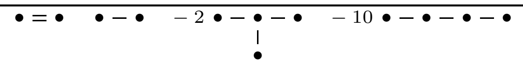
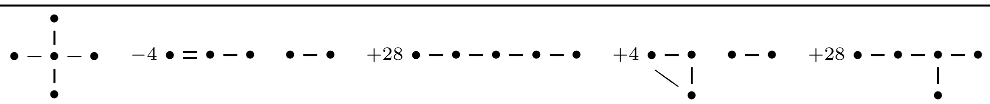
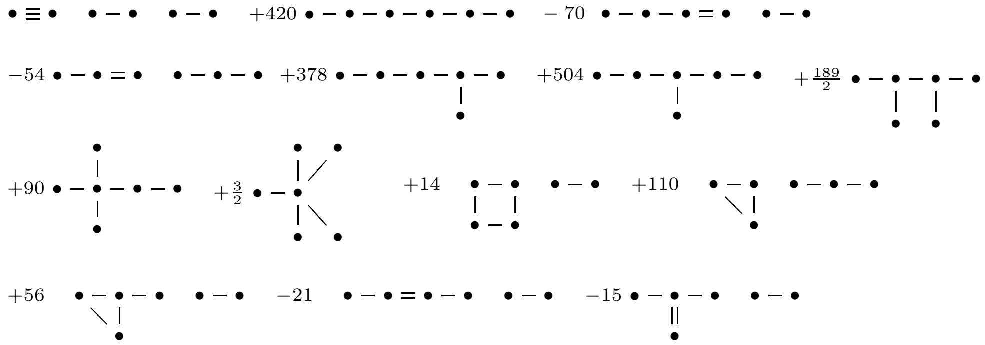
The last equality is due to the fact that $R_i(P)$ satisfies equation (11).
Similarly, $B_{P,\sigma}B_{Q,\sigma}$ is in the kernel of R_n. Hence $(\mathbf{z}\cdot\mathbf{y} - \mathbf{t}\cdot\mathbf{x})\,[R_n(PQ)]\,(\mathbf{x}+\mathbf{z},\ \mathbf{y}+\mathbf{t}) = 0.$■

So Proposition 28 gives a way to construct solutions of equation (11) for B_n from already known solutions of this equation for B_m with $m < n$, by taking the disjoint unions of the solution graphs. In fact, according to Proposition 26, if a graph is a solution for B_m, then it is also a solution for B_n with $n > m$.
Thus we may formulate the following conjecture :

Conjecture 29
For every integer $n \geq 2$, there exists a unique polynomial of degree $4(n-1)$ in $S^W_{\mathfrak{sl}_2}$ (i.e. a linear combination of

graphs with $n-1$ edges) which is solution of equation (11). This polynomial is represented by a graph which is made up with the linear graph without loops and without isolated vertices, with $n-1$ edges and n vertices, and with other graphs which may be seen as corrective terms, and which all have $n-1$ edges and n (non-isolated) vertices.
This graph is called the $n-$th simple graph.

By some Maple calculations, we determine the list of the first simple graphs :

B_n	$n-$th simple graph
B_2	$\bullet - \bullet$
B_3	$\bullet - \bullet - \bullet$
B_4	
B_5	
B_6	

The number of graphs that we may construct from the simple graphs by using Proposition 28 equals the number of linear graphs without loops and without isolated vertices of Proposition 24, since the graphs constructed like this are obtained by substituting, to certain linear graphs without loops and without isolated vertices, a linear combination of graphs containing this graph and other graphs (possibly non-linear or with loops), but having the same number of vertices and the same number of edges.

Conjecture 30
For every integer $n \geq 2$, the $\pi(n)$ graphs constructed according to the process defined above are the only solutions of equation (11).

As for D_n, we may write, with the help of Maple and of Proposition 17, the solutions of degree lower than $4(n-1)$ of equation (11) for $n \in \{2, 3, 4\}$. We conjecture that this equation has no solution of degree strictly higher than $4(n-1)$.

D_n	Solutions of equation (11)		
D_2	1		
D_3	1		
D_4	1	$\bullet - \bullet$	$\bullet - \bullet \quad \bullet - \bullet \quad -2 \bullet - \bullet - \bullet$

3 Study of $B_2 - C_2$, $D_2 = A_1 \times A_1$, $B_3 - C_3$, and $D_3 = A_3$

Let us now use the results of the general case previously established by applying them to the cases of B_2, D_2, B_3 and D_3. It will enable us to calculate explicitly the dimension of their $0-$th space of Poisson homology, and so to treat entirely the rank 3, and at the same time to verify the conjecture of J. Alev for this rank.

3.1 Study of B_2 and D_2

The computation of the dimension of the $0-$th space of Poisson homology in the case of B_2 was made by J. Alev and L. Foissy in [AF06]. Here, we rediscover this result by another method. We will then deduce the dimension of the 0−th space of Poisson homology in the case of D_2.

Proposition 31
For B_2, the space of solutions of equation (11) is the plane generated by the polynomials 1 *and* $(x_1y_2 - y_1x_2)^2$. *So, the dimension of the* $0-th$ *space of Poisson homology of* B_2 *is* $\boxed{\dim(HP_0(B_2)) = 2}$.
The elements 1 *and* $(x_1y_2 - y_1x_2)^2$ *do not belong to* $\{S^W,\ S^W\}$.

Proof :
• Let $X = x_1y_2 - y_1x_2$. We have $\mathbb{C}[X] \subset S_{\mathfrak{sl}_2}$. Now, according to Proposition 7, we have $\dim\left(S(2l)_{\mathfrak{sl}_2}\right) = 1$ therefore the Poincaré series of $S_{\mathfrak{sl}_2}$ is $\frac{1}{1-z^2}$. This series is exactly the Poincaré series of $\mathbb{C}[X]$, so $S_{\mathfrak{sl}_2} = \mathbb{C}[X]$. Consequently, $S_{\mathfrak{sl}_2}^W = \mathbb{C}[X]^W = R_2(\mathbb{C}[X]) = \mathbb{C}[X^2]$.
• According to Proposition 4, the elements 1 and X^2 do not belong to $\{S^W,\ S^W\}$. We show by a direct calculation (for example with Maple ; see section 4) that the element X^2 satisfies equation (11).
• In order to have the asserted result, it is sufficient, according to Corollary 13, to show that for every $j \in \mathbb{N}\backslash\{0,\ 1\}$, the element X^{2j} does not satisfy equation (11), i.e. $E_2(X^{2j}) \neq 0$. To do this, it is sufficient, according to remark 15, to show that $E_2'(X^{2j}) \neq 0$. From the expression

$$E_{int}^2(X^{2j}) = z_1R_2\big(y_1((x_1+z_1)y_2 - (y_1+t_1)(x_2+z_2))^{2j}\big) + z_2R_2\big(y_2((x_1+z_1)y_2 - (y_1+t_1)(x_2+z_2))^{2j}\big),$$

we deduce

$$E_2'(X^{2j}) = \frac{1}{8}\,z_1 \sum_{sign\ changes} y_1(-(y_1+t_1)z_2)^{2j} + \frac{1}{8}\,z_2 \sum_{sign\ changes} y_1(z_1y_1 - t_1z_2)^{2j},$$

i.e.

$$E_2'(X^{2j}) = \tfrac{1}{4}\big[z_1y_1z_2^{2j}(y_1+t_1)^{2j} - z_1y_1z_2^{2j}(-y_1+t_1)^{2j} + z_2y_1(z_1y_1 - t_1z_2)^{2j} - z_2y_1(-z_1y_1 - t_1z_2)^{2j}\big].$$

So the term of the type $\alpha z_1t_1y_1^{2j}z_2^{2j}$ is $\frac{1}{4}z_1y_1z_2^{2j}(C_{2j}^1y_1^{2j-1}t_1) - \frac{1}{4}z_1y_1z_2^{2j}(-C_{2j}^1y_1^{2j-1}t_1)$, and we find $\alpha = j \neq 0$. This shows that X^{2j} does not satisfy equation (11).■

Proposition 32
For $D_2 = A_1 \times A_1$, the space of the solutions of equation (11) is the one-dimensional space generated by 1. *So the dimension of the* $0-th$ *space of Poisson homology of* D_2 *is* 1, *i.e.* $\boxed{\dim(HP_0(D_2)) = 1}$.
The element 1 *is the only element which does not belong to* $\{S^W,\ S^W\}$.

Proof :
The method is the same as the one for B_2 : letting $X = x_1y_2 - y_1x_2$, we have $S_{\mathfrak{sl}_2}^W = \mathbb{C}[X^2]$. According to Proposition 4, the element 1 does not belong to $\{S^W,\ S^W\}$. We show by a calculation (for example with Maple ; see section 4) that the element X^2 does not satisfy equation (11).
So it is sufficient, according to Corollary 13, to show that for every $j \in \mathbb{N}\backslash\{0,\ 1\}$, $E_2'(X^{2j}) \neq 0$. Now this expression is the same as the one obtained for B_2 ; therefore it is not zero.■

3.2 Study of B_3 - Vectors of highest weight 0

In order to study B_3, we begin making by making the vectors of highest weight 0 explicit.

Proposition 33
We set $X = x_1y_2 - y_1x_2,\ Y = x_2y_3 - y_2x_3$ *and* $Z = x_3y_1 - y_3x_1$.
Then $S_{\mathfrak{sl}_2}^{B_3}$ *is the image of* $\mathbb{C}[X,\ Y,\ Z]$ *by* R_3.

Proof : according to Proposition 8, $\mathbb{C}[X,\ Y,\ Z] = S_{\mathfrak{sl}_2}$, thus $\mathbb{C}[X,\ Y,\ Z]^{B_3} = S_{\mathfrak{sl}_2}^{B_3}$.

Remark 34
The only monomials $X^iY^jZ^k$ $(i,\ j,\ k) \in \mathbb{N}^3$ *of which the image by* R_3 *is not zero are*
▷ *the monomials of the form* $X^iY^jZ^k$ *with* $i,\ j,\ k$ *even,*
▷ *the monomials of the form* $X^iY^jZ^k$ *with* $i,\ j,\ k$ *odd and all distinct. For these monomials, it is even sufficient to take* $i < j < k$.
So the vector of highest weigh 0 *of smallest degree not multiple of* 4 *has degree* $2(1+3+5) = 18$.

Proposition 35
The elements 1 *and* $R_3(X^2)$ *do not belong to* $\{S^W,\ S^W\}$.
The elements 1, $R_3(X^2)$ *and* $R_3(X^2Y^2)$ *satisfy equation (11).*
So, $\dim(HP_0(B_3)) \geq 3$.

Proof :
We know according to Proposition 4 that the elements 1 and $R_3(X^2)$ do not belong to $\{S^W,\ S^W\}$.
To show that the elements $R_3(X^2)$ and $R_3(X^2Y^2)$ satisfy equation (11), we make a calculation with Maple : see section 4.■

In order to prove the equality $\dim(HP_0(B_3)) = 3$, we will show that the elements of $S_{\mathfrak{sl}_2}^{B_3}$ which do not belong to the space $\langle 1,\ R_3(X^2),\ R_3(X^2Y^2)\rangle$ do not satisfy equation (11). It is sufficient, according to remark 15, to show that these elements do not satisfy equation (17). It is the aim of the following section.

3.3 Study of B_3 - Equation of Berest-Etingof-Ginzburg

To each polynomial of the form $P := R_3(X^iY^jZ^k)$ specified in remark 34, we associate a monomial M_P which appears with a non-zero coefficient in equation (17), and such that if P_1 and P_2 are two distinct polynomials, then the monomial M_{P_1} does not appear in $E'_n(P_2)$ and the monomial M_{P_2} does not appear in $E'_n(P_1)$, where $E'_n(P)$ denotes equation (17).

To the polynomial $R_3(X^{2j})$ (with $j \geq 2$), we associate the monomial $M_j = z_1t_1y_1^{2j}z_3^{2j}$.

To the polynomial $R_3(X^{2j}Y^{2l})$ (with $1 \leq l \leq j$ and $j \geq 2$), we associate $M_{j,l} = z_1t_1y_1^{2j+2l}z_3^{2j}z_2^{2l}$.

To the polynomial $R_3(X^{2j}Y^{2l}Z^{2k})$ (with $1 \leq l \leq k \leq j$), we associate $M_{j,k,l} = z_1t_1y_1^{2j+2l}z_3^{2j+2k}z_2^2t_1^{2k}z_1^{2l-2}$.

To the polynomial $R_3(X^{2i+1}Y^{2j+1}Z^{2k+1})$ (with $0 \leq i < j < k$), we associate $\widetilde{M}_{k,j,i} = z_1t_1y_1^{2i+2k+2}z_3^{2j+2k+2}z_2^2t_1^{2j+1}z_1^{2i-1}$.

3.3.1 First step

For every polynomial P, we calculate the coefficient of the monomial M_P which appears in equation $E'_3(P)$.

Case 1. $P = R_3(X^{2j}) = \frac{1}{3}(X^{2j} + Y^{2j} + Z^{2j})$ (with $j \geq 2$) :
We have
$|W|E^3_{int}(P) = \frac{|W|}{3}R_3\Big[(z_1y_1 + z_2y_2 + z_3y_3)\big[((z_1y_2 - (y_1+t_1)z_2)^{2j} + (z_2y_3 - y_2z_3)^{2j} + (z_3(y_1+t_1) - y_3z_1)^{2j}\big]\Big]$,
Thus

$$|W|E^3_{int}(P) = y_2(\dots) + y_3(\dots) + \frac{2}{3}\sum_{sign\ changes} z_1y_1\big[((y_1+t_1)z_2)^{2j} + (z_3(y_1+t_1))^{2j}\big]$$
$$+\frac{2}{3}\sum_{sign\ changes} z_2y_1\big[(z_1y_1 - t_1z_2)^{2j} + (y_1z_3)^{2j} + (t_1z_3)^{2j}\big] + \frac{2}{3}\sum_{sign\ changes} z_3y_1\big[(t_1z_2)^{2j} + (z_2y_1)^{2j} + (z_3t_1 - y_1z_1)^{2j}\big].$$

Therefore,

$$\begin{aligned}
|W|E'_3(P) \;=\;& \tfrac{8}{3}\Big[z_1y_1\big[((y_1+t_1)z_2)^{2j} + (z_3(y_1+t_1))^{2j}\big] - z_1y_1\big[((-y_1+t_1)z_2)^{2j} + (z_3(-y_1+t_1))^{2j}\big]\Big]\\
&+\tfrac{8}{3}\Big[z_2y_1\big[(z_1y_1 - t_1z_2)^{2j} + \underline{(y_1z_3)^{2j}} + \underline{(t_1z_3)^{2j}}\big] - z_2y_1\big[(-z_1y_1 - t_1z_2)^{2j} + \underline{(y_1z_3)^{2j}} + \underline{(t_1z_3)^{2j}}\big]\Big]\\
&+\tfrac{8}{3}\Big[z_3y_1\big[\underline{(t_1z_2)^{2j}} + \underline{(z_2y_1)^{2j}} + (z_3t_1 - y_1z_1)^{2j}\big] - z_3y_1\big[\underline{(t_1z_2)^{2j}} + \underline{(z_2y_1)^{2j}} + (z_3t_1 + y_1z_1)^{2j}\big]\Big],
\end{aligned} \tag{18}$$

where the underlined terms cancel out.
The monomial $M_j = z_1t_1y_1^{2j}z_3^{2j}$ appears only in the first line of the last expression, and its coefficient is $\frac{1}{|W|}\frac{8}{3}\big[2C^1_{2j} + 2C^1_{2j}\big] = \frac{32}{3}j$.
So, $\boxed{\text{the coefficient of } M_j \text{ in } E'_3(P) \text{ is } \frac{2}{9}j}$.

Case 2. $P = R_3(X^{2j}Y^{2l})$ (with $1 \leq l \leq j$ and $j \geq 2$) :

We have

$$|W|E^3_{int}(P) = \frac{|W|}{6}R_3\Big[(z_1y_1 + z_2y_2 + z_3y_3)$$
$$\big[((z_1y_2 - (y_1+t_1)z_2)^{2j}(z_2y_3 - y_2z_3)^{2l} + (z_2y_3 - y_2z_3)^{2j}(z_3(y_1+t_1) - y_3z_1)^{2l} + (z_3(y_1+t_1) - y_3z_1)^{2j}(z_1y_2 - (y_1+t_1)z_2)^{2l}$$
$$((z_1y_2 - (y_1+t_1)z_2)^{2l}(z_2y_3 - y_2z_3)^{2j} + (z_2y_3 - y_2z_3)^{2l}(z_3(y_1+t_1) - y_3z_1)^{2j} + (z_3(y_1+t_1) - y_3z_1)^{2l}(z_1y_2 - (y_1+t_1)z_2)^{2j}\big]\Big],$$

Thus
$$
\begin{aligned}
|W|E_3'(P) \;=\; & \tfrac{8}{6}\Big[z_1y_1\big[(z_3(y_1+t_1))^{2j}((y_1+t_1)z_2)^{2l}\big] - z_1y_1\big[(z_3(-y_1+t_1))^{2j}((-y_1+t_1)z_2)^{2l}\big] \\
& +z_2y_1\big[(z_1y_1-t_1z_2)^{2j}(y_1z_3)^{2l} + \underline{(y_1z_3)^{2j}(z_3t_1)^{2l}} + (z_3t_1)^{2j}(z_1y_1-t_1z_2)^{2l}\big] - \dots \\
& +z_3y_1\big[\underline{(t_1z_2)^{2j}(z_2y_1)^{2l}} + (z_2y_1)^{2j}(z_3t_1-y_1z_1)^{2l} + (z_3t_1-y_1z_1)^{2j}(t_1z_2)^{2l}\big] - \dots \\
& +z_1y_1\big[(z_3(y_1+t_1))^{2l}((y_1+t_1)z_2)^{2j}\big] - z_1y_1\big[(z_3(-y_1+t_1))^{2l}((-y_1+t_1)z_2)^{2j}\big] \\
& +z_2y_1\big[(z_1y_1-t_1z_2)^{2l}(y_1z_3)^{2j} + \underline{(y_1z_3)^{2l}(z_3t_1)^{2j}} + (z_3t_1)^{2l}(z_1y_1-t_1z_2)^{2j}\big] - \dots \\
& +z_3y_1\big[\underline{(t_1z_2)^{2l}(z_2y_1)^{2j}} + (z_2y_1)^{2l}(z_3t_1-y_1z_1)^{2j} + (z_3t_1-y_1z_1)^{2l}(t_1z_2)^{2j}\big] - \dots\Big],
\end{aligned}
\tag{19}
$$

where on each line, the suspension points stand for the image of the expression by the sign change $y_1 \mapsto -y_1$. As in the preceding case, the underlined terms cancel out.

▷ If $j \neq l$ and $l \neq 1$, the monomial $M_{j,l}$ appears only in the first line of the preceding expression and its coefficient in this expression is $\frac{32}{6}(j+l)$.

▷ If $j \neq l$ and $l = 1$, the monomial $M_{j,l}$ also appears in the 5−th line of (19) and its coefficient is $\frac{32}{6}(j+l-l)$.

▷ If $j = l$, we have $l \neq 1$ (because $j \geq 2$). Then $M_{j,l}$ also appears in the fourth line and its coefficient is thus $\frac{64}{6}(j+l)$.

The following table collects the coefficients of the monomial $M_{j,l}$ in $E_3'(P)$.

$(j,\,l)$	$j \neq l$ and $l \neq 1$	$j \neq l$ and $l = 1$	$j = l$
Coefficient	$(j+l)/9$	$j/9$	$4j/9$

Case 3. $P = R_3(X^{2j}Y^{2l}Z^{2k})$ (with $1 \leq l \leq k \leq j$) :

we have

$$|W|E^3_{int}(P) = \frac{|W|}{6}R_3\Big[(z_1y_1+z_2y_2+z_3y_3)\big[((z_1y_2-(y_1+t_1)z_2)^{2j}(z_2y_3-y_2z_3)^{2l}(z_3(y_1+t_1)-y_3z_1)^{2k} + \sum_{\substack{permutations\\ of\ (j,l,k)}} \dots\big]\Big],$$

So
$$
\begin{aligned}
|W|E_3'(P) = \tfrac{8}{6}\sum_{\substack{permutations\\ of\ (j,l,k)}} & \Big[z_2y_1(z_1y_1-t_1z_2)^{2j}(y_1z_3)^{2l}(z_3t_1)^{2k} - z_2y_1(-z_1y_1-t_1z_2)^{2j}(y_1z_3)^{2l}(z_3t_1)^{2k} \\
& +z_3y_1(t_1z_2)^{2j}(z_2y_1)^{2l}(z_3t_1-y_1z_1)^{2k} - z_3y_1(t_1z_2)^{2j}(z_2y_1)^{2l}(z_3t_1+y_1z_1)^{2k}\Big],
\end{aligned}
$$

We denote by $\alpha_{j,k,l}$ the coefficient of the monomial $M_{j,k,l}$ in the preceding expression, and we distinguish 4 cases :

▷ If $l = k = j$, we have $M_{j,j,j} = z_1t_1z_1^{2j-2}y_1^{4j}z_3^{4j}z_2^2t_1^{2j}$ and $\alpha_{j,j,j} = -32j$.

▷ If $l = k < j$, we have $M_{j,l,l} = z_1t_1z_1^{2l-2}y_1^{2j+2l}z_3^{2j+2l}z_2^2t_1^{2l}$, and
$$
\begin{aligned}
|W|E_3'(P) = \tfrac{8}{6}\Big[& z_2y_1(z_1y_1-t_1z_2)^{2j}(y_1z_3)^{2l}(z_3t_1)^{2l} - \dots + z_3y_1(t_1z_2)^{2j}(z_2y_1)^{2l}(z_3t_1-y_1z_1)^{2l} - \dots \\
& +z_2y_1(z_1y_1-t_1z_2)^{2l}(y_1z_3)^{2j}(z_3t_1)^{2l} - \dots + z_3y_1(t_1z_2)^{2l}(z_2y_1)^{2j}(z_3t_1-y_1z_1)^{2l} - \dots \\
& +z_2y_1(z_1y_1-t_1z_2)^{2l}(y_1z_3)^{2l}(z_3t_1)^{2j} - \dots + z_3y_1(t_1z_2)^{2l}(z_2y_1)^{2l}(z_3t_1-y_1z_1)^{2j} - \dots\Big],
\end{aligned}
$$
where on each line, the suspension points stand for the image of the expression by the sign change $y_1 \mapsto -y_1$. So the searched coefficient is $\alpha_{j,l,l} = -\frac{32}{3}l$.

▷ If $l < k = j$, we have $M_{j,j,l} = z_1t_1z_1^{2l-2}y_1^{2j+2l}z_3^{4j}z_2^2t_1^{2j}$. Similarly, we find $\alpha_{j,j,l} = -32j$.

▷ If $l < k < j$, we have

$$|W|E_3'(P) = \frac{8}{6}\Big[z_2y_1(z_1y_1-t_1z_2)^{2j}(y_1z_3)^{2l}(z_3t_1)^{2k} - \dots \\ +z_3y_1(t_1z_2)^{2j}(z_2y_1)^{2l}(z_3t_1-y_1z_1)^{2k} - \dots \\ +z_2y_1(z_1y_1-t_1z_2)^{2j}(y_1z_3)^{2k}(z_3t_1)^{2l} - \dots \\ +z_3y_1(t_1z_2)^{2j}(z_2y_1)^{2k}(z_3t_1-y_1z_1)^{2l} - \dots \\ +z_2y_1(z_1y_1-t_1z_2)^{2k}(y_1z_3)^{2j}(z_3t_1)^{2l} - \dots \\ +z_3y_1(t_1z_2)^{2k}(z_2y_1)^{2j}(z_3t_1-y_1z_1)^{2l} - \dots \\ +z_2y_1(z_1y_1-t_1z_2)^{2l}(y_1z_3)^{2j}(z_3t_1)^{2k} - \dots \\ +z_3y_1(t_1z_2)^{2l}(z_2y_1)^{2j}(z_3t_1-y_1z_1)^{2k} - \dots \\ +z_2y_1(z_1y_1-t_1z_2)^{2k}(y_1z_3)^{2l}(z_3t_1)^{2j} - \dots \\ +z_3y_1(t_1z_2)^{2k}(z_2y_1)^{2l}(z_3t_1-y_1z_1)^{2j} - \dots \\ +z_2y_1(z_1y_1-t_1z_2)^{2l}(y_1z_3)^{2k}(z_3t_1)^{2j} - \dots \\ +z_3y_1(t_1z_2)^{2l}(z_2y_1)^{2k}(z_3t_1-y_1z_1)^{2j} - \dots\Big], \tag{20}$$

where, again, the suspension points stand for the image of the expression by the sign change $y_1 \mapsto -y_1$. The searched monomial appears only on the 7−th line, and its coefficient is $\alpha_{j,k,l} = -\frac{32}{6}l$.

Finally we collect in the following table the coefficients of the monomial $M_{j,k,l}$ in $E_3'(P)$.

$(j,\,k,\,l)$	$l=k=j$	$l=k<j$	$l<k=j$	$l<k<j$
Coefficient	$-2l/3$	$-2l/9$	$-2l/9$	$-l/9$

Case 4. $P = R_3(X^{2i+1}Y^{2j+1}Z^{2k+1})$ (with $0 \le i < j < k$) : we have

$$|W|E^3_{int}(P) = \frac{|W|}{6}R_3\Big[(z_1y_1+z_2y_2+z_3y_3)\big[((z_1y_2-(y_1+t_1)z_2)^{2i+1}(z_2y_3-y_2z_3)^{2j+1}(z_3(y_1+t_1)-y_3z_1)^{2k+1} + \sum_{\substack{permutations\\ of\ (i,j,k)}} \dots\big]\Big],$$

So
$$|W|E_3'(P) = \frac{8}{6}\sum_{\substack{permutations\\ of\ (j,j,k)}}\Big[z_2y_1(z_1y_1-t_1z_2)^{2i+1}(y_1z_3)^{2j+1}(z_3t_1)^{2k+1} + z_2y_1(-z_1y_1-t_1z_2)^{2i+1}(y_1z_3)^{2j+1}(z_3t_1)^{2k+1} \\ +z_3y_1(t_1z_2)^{2i+1}(z_2y_1)^{2j+1}(z_3t_1-y_1z_1)^{2k+1} + z_3y_1(t_1z_2)^{2i+1}(z_2y_1)^{2j+1}(z_3t_1+y_1z_1)^{2k+1}\Big] \\ = z_2y_1(z_1y_1-t_1z_2)^{2i+1}(y_1z_3)^{2j+1}(z_3t_1)^{2k+1} + z_2y_1(-z_1y_1-t_1z_2)^{2i+1}(y_1z_3)^{2j+1}(z_3t_1)^{2k+1} \\ +z_3y_1(t_1z_2)^{2i+1}(z_2y_1)^{2j+1}(z_3t_1-y_1z_1)^{2k+1} + z_3y_1(t_1z_2)^{2i+1}(z_2y_1)^{2j+1}(z_3t_1+y_1z_1)^{2k+1} \\ +z_2y_1(z_1y_1-t_1z_2)^{2i+1}(y_1z_3)^{2k+1}(z_3t_1)^{2j+1} + z_2y_1(-z_1y_1-t_1z_2)^{2i+1}(y_1z_3)^{2k+1}(z_3t_1)^{2j+1} \\ +z_3y_1(t_1z_2)^{2i+1}(z_2y_1)^{2k+1}(z_3t_1-y_1z_1)^{2j+1} + z_3y_1(t_1z_2)^{2i+1}(z_2y_1)^{2k+1}(z_3t_1+y_1z_1)^{2j+1} \\ +z_2y_1(z_1y_1-t_1z_2)^{2j+1}(y_1z_3)^{2i+1}(z_3t_1)^{2k+1} + z_2y_1(-z_1y_1-t_1z_2)^{2j+1}(y_1z_3)^{2i+1}(z_3t_1)^{2k+1} \\ +z_3y_1(t_1z_2)^{2j+1}(z_2y_1)^{2i+1}(z_3t_1-y_1z_1)^{2k+1} + z_3y_1(t_1z_2)^{2j+1}(z_2y_1)^{2i+1}(z_3t_1+y_1z_1)^{2k+1} \\ +z_2y_1(z_1y_1-t_1z_2)^{2k+1}(y_1z_3)^{2j+1}(z_3t_1)^{2i+1} + z_2y_1(-z_1y_1-t_1z_2)^{2k+1}(y_1z_3)^{2j+1}(z_3t_1)^{2i+1} \\ +z_3y_1(t_1z_2)^{2k+1}(z_2y_1)^{2j+1}(z_3t_1-y_1z_1)^{2i+1} + z_3y_1(t_1z_2)^{2k+1}(z_2y_1)^{2j+1}(z_3t_1+y_1z_1)^{2i+1} \\ +z_2y_1(z_1y_1-t_1z_2)^{2j+1}(y_1z_3)^{2k+1}(z_3t_1)^{2i+1} + z_2y_1(-z_1y_1-t_1z_2)^{2j+1}(y_1z_3)^{2k+1}(z_3t_1)^{2i+1} \\ +z_3y_1(t_1z_2)^{2j+1}(z_2y_1)^{2k+1}(z_3t_1-y_1z_1)^{2i+1} + z_3y_1(t_1z_2)^{2j+1}(z_2y_1)^{2k+1}(z_3t_1+y_1z_1)^{2i+1} \\ +z_2y_1(z_1y_1-t_1z_2)^{2k+1}(y_1z_3)^{2i+1}(z_3t_1)^{2j+1} + z_2y_1(-z_1y_1-t_1z_2)^{2k+1}(y_1z_3)^{2i+1}(z_3t_1)^{2j+1} \\ +z_3y_1(t_1z_2)^{2k+1}(z_2y_1)^{2i+1}(z_3t_1-y_1z_1)^{2j+1} + z_3y_1(t_1z_2)^{2k+1}(z_2y_1)^{2i+1}(z_3t_1+y_1z_1)^{2j+1} \tag{21}$$

In order to find the monomials of the form $\alpha\widetilde{M}_{k,j,i}$, we are first interested in the exponent of z_3, then in the one of y_1. So it remains only the lines 1 and 3, then only the line 3. The coefficient of $\widetilde{M}_{k,j,i}$ in $|W|E_3'(P)$ is $-\frac{16}{6}(2i+1)$. So, $\boxed{\text{the coefficient of } \widetilde{M}_{k,j,i} \text{ in } E_3'(P) \text{ is } -\frac{1}{18}(2i+1)}$.

3.3.2 Second step

For two distinct polynomials P_1 and P_2, of the same degree greater or equal to 12, we show that M_{P_1} does not appear in $E'(P_2)$.

Case 1. $P = R_3(X^{2j})$ $(j \ge 3)$: according to (18), no term of $E_3'(P)$ contains at the same time $z_1,\ z_2,\ z_3$. Now if $l+k=j$ with $1 \le k \le l$ and $l \ge 2$, then $M_{l,k} = z_1t_1y_1^{2l+2k}z_3^{2l}z_2^{2k}$, therefore $M_{l,k}$ does not appear in $E_3'(P)$. Similarly, if $m+l+k=j$ with $0 < k \le l \le m$, we have $M_{m,l,k} = z_1t_1y_1^{2m+2k}z_3^{2m+2l}z_2^2t_1^{2l}z_1^{2k-2}$, therefore $M_{m,l,k}$

does not appear in $E'_3(P)$.

Case 2. $P = R_3(X^{2j}Y^{2l})$ (with $1 \leq l \leq j$ and $j \geq 2$) : equation (19) may be written

$$\begin{array}{rcl}\frac{6}{8}|W|E'_3(P) & = & z_1y_1(z_3(y_1+t_1))^{2j}((y_1+t_1)z_2)^{2l} - z_1y_1(z_3(-y_1+t_1))^{2j}((-y_1+t_1)z_2)^{2l} \\ & & +z_1y_1(z_3(y_1+t_1))^{2l}((y_1+t_1)z_2)^{2j} - z_1y_1(z_3(-y_1+t_1))^{2l}((-y_1+t_1)z_2)^{2j} \\ & & +z_2y_1(z_1y_1-t_1z_2)^{2j}(y_1z_3)^{2l} + z_2y_1(z_3t_1)^{2j}(z_1y_1-t_1z_2)^{2l} \\ & & -z_2y_1(-z_1y_1-t_1z_2)^{2j}(y_1z_3)^{2l} - z_2y_1(z_3t_1)^{2j}(-z_1y_1-t_1z_2)^{2l} \\ & & +z_2y_1(z_1y_1-t_1z_2)^{2l}(y_1z_3)^{2j} + z_2y_1(z_3t_1)^{2l}(z_1y_1-t_1z_2)^{2j} \\ & & -z_2y_1(-z_1y_1-t_1z_2)^{2l}(y_1z_3)^{2j} - z_2y_1(z_3t_1)^{2l}(-z_1y_1-t_1z_2)^{2j} \\ & & +z_3y_1(z_2y_1)^{2j}(z_3t_1-y_1z_1)^{2l} + z_3y_1(z_3t_1-y_1z_1)^{2j}(t_1z_2)^{2l} \\ & & -z_3y_1(z_2y_1)^{2j}(z_3t_1+y_1z_1)^{2l} - z_3y_1(z_3t_1+y_1z_1)^{2j}(t_1z_2)^{2l} \\ & & +z_3y_1(z_2y_1)^{2l}(z_3t_1-y_1z_1)^{2j} + z_3y_1(z_3t_1-y_1z_1)^{2l}(t_1z_2)^{2j} \\ & & -z_3y_1(z_2y_1)^{2l}(z_3t_1+y_1z_1)^{2j} - z_3y_1(z_3t_1+y_1z_1)^{2l}(t_1z_2)^{2j}.\end{array} \qquad (22)$$

• If $m = j + l$, then $M_m = z_1t_1y_1^{2m}z_3^{2m}$ does not appear in $E(P)$. Indeed, the largest exponent of z_3 in $E(P)$ is $\leq 2j+1 < 2m$.

• If $m + p = j + l$, with $1 \leq p \leq m$, $m \geq 2$, $m \neq j$ and $p \neq l$, then $M_{m,p} = z_1t_1y_1^{2m+2p}z_3^{2m}z_2^{2p}$ does not appear in $E(P)$. In fact,
▷ if $m > j$, necessary $p < l$. Then the largest exponent of z_3 in $E(P)$ is $\leq 2j+1 < 2m$.
▷ if $m < j$, necessary $p > l$. Then the only terms of $E(P)$ which have z_1t_1 with the exponent 1 are

$$\begin{array}{rcl} L_1 & = & 8jz_1t_1y_1^{2j+2l}z_3^{2j}z_2^{2l} \\ L_2 & = & 8lz_1t_1y_1^{2j+2l}z_3^{2l}z_2^{2j} \\ L_3 & = & -4z_1t_1y_1^{2j+2}z_3^{2j}z_2^{2} \;\text{ if } l=1 \\ L_4 & = & -4z_1t_1y_1^{2j+2}z_3^{2}z_2^{2j} \;\text{ if } l=1. \end{array}$$

L_1 and L_3 are not colinear to $M_{m,p}$, because they are colinear to $M_{j,l}$.
L_2 is colinear to $M_{m,p}$ only if $m = l$ and $j = p$. But then $m = l \leq j = p \leq m$, hence $m = j$ and $p = l$, which is absurd. The same argument may be applied to L_4.

• If $m+p+q = j+l$, with $1 \leq q \leq p \leq m$, then $M_{m,p,q} = z_1t_1y_1^{2m+2q}z_3^{2m+2p}z_2^2t_1^{2p}z_1^{2q-2}$ does not appear in $E(P)$. In fact,
▷ If $m + p > j$, the greatest exponent of z_3 in $E(P)$ is $\leq 2j+1 < 2m+2p$.
▷ If $m + p < j$, we have necessarily $q \geq l+1$, thus $2q-1 > 2l \geq 2$. Then the only terms of $E(P)$ which have z_1 with the exponent $2q-1$ are

$$\begin{array}{rcl} L_1 & = & -2C_{2j}^{2q-1}y_1^{2q+2l}z_1^{2q-1}z_2^{2j-2q+2}z_3^{2l}t_1^{2j-2q+1} \\ L_2 & = & -2C_{2j}^{2q-1}y_1^{2q}z_1^{2q-1}z_2^{2j-2q+2}z_3^{2l}t_1^{2j-2q+2l+1} \\ L_3 & = & -2C_{2j}^{2q-1}y_1^{2q}z_1^{2q-1}z_2^{2l}z_3^{2j-2q+2}t_1^{2j-2q+2l-1} \\ L_4 & = & -2C_{2j}^{2q-1}y_1^{2q+2l}z_1^{2q-1}z_2^{2l}z_3^{2j-2q+2}t_1^{2j-2q+1}. \end{array}$$

If L_1 is colinear to $M_{m,p,q}$, we have $\left\{\begin{array}{rcl} m+q & = & q+l \\ 2 & = & 2j-2q+2 \\ 2m+2p & = & 2l \\ 2p+1 & = & 2j-2q+1 \end{array}\right.$, hence $p = 0$, which is absurd.
Similarly, L_2, L_3 and L_4 are not colinear to $M_{m,p,q}$.
▷ If $m + p = j$, we have necessarily $q = l$ (but only $2q-1 \geq 1$). Then we have two cases :
◇ If $l \geq 2$, i.e. $q \geq 2$, in addition to the terms $L_1, \ldots, L_4$, we obtain the terms

$$\begin{array}{rcl} L_5 & = & -4lz_2y_1z_3^{2j}t_1^{2j}z_1^{2l-1}y_1^{2l-1}t_1z_2 \\ L_6 & = & -4lz_2y_1z_1^{2l-1}y_1^{2l-1}t_1z_2y_1^{2j}z_3^{2j} \\ L_7 & = & -4lz_3y_1z_2^{2j}y_1^{2j}z_3t_1y_1^{2l-1}z_1^{2l-1} \\ L_8 & = & -4lz_3y_1z_3t_1y_1^{2l-1}z_1^{2l-1}t_1^{2j}z_2^{2j}. \end{array}$$

If L_5 is colinear to $M_{m,p,q}$, we have in particular $2m + 2l = 2l$, hence $m = 0$, which is absurd.
Similarly, L_6, L_7 and L_8 are not colinear to $M_{m,p,q}$.
◇ If $l = 1$, i.e. $q = 1$, in addition to the terms $L_1, \ldots, L_8$, we also obtain the terms

$$\begin{array}{c} z_1y_1z_3^{2j}z_2^2(y_1+t_1)^{2j+2} - z_1y_1z_3^{2j}z_2^2(-y_1+t_1)^{2j+2} \\ z_1y_1z_3^2z_2^{2j}(y_1+t_1)^{2j+2} - z_1y_1z_3^2z_2^{2j}(-y_1+t_1)^{2j+2} \end{array}$$

Among these terms, the ones which have t_1 with the exponent $2p$ vanish because of the signs in the expansion of the binomial.

Case 3. $P = R_3(X^{2j}Y^{2l}Z^{2k})$ (with $1 \leq l \leq k \leq j$) :
• According to formula (20), the greatest exponent of z_3 in $E_3'(P)$ is $\leq 2l + 2k$. So if $m = j + k + l$, then $M_m = z_1 t_1 y_1^{2m} z_3^{2m}$ does not appear in $E_3'(P)$.

• If $m + p = j + l + k$ with $1 \leq p \leq m$, $m \geq 2$, then $M_{m,p} = z_1 t_1 y_1^{2m+2p} z_3^{2m} z_2^{2p}$ doesn't appear in $E_3'(P)$. In fact,
$\triangleright$ If $j + k < m$, then z_3 never appears in $E_3'(P)$ with the exponent $2m$.
$\triangleright$ If $j + k \geq m$, we write the terms of $E_3'(P)$ which have y_1 with the exponent $2m + 2p$:

$$\begin{array}{lcl|lcl}
L_1 &=& -4jz_2y_1z_1^{2j-1}y_1^{2j-1}t_1z_2y_1^{2l}z_3^{2l+2k}t_1^{2k} & L_7 &=& -4lz_2y_1z_1^{2l-1}y_1^{2l-1}t_1z_2y_1^{2j}z_3^{2j+2k}t_1^{2k} \\
L_2 &=& -4kz_3y_1t_1^{2j}z_2^{2j+2l}y_1^{2l}z_3t_1y_1^{2k-1}z_1^{2k-1} & L_8 &=& -4kz_3y_1t_1^{2l}z_2^{2l+2j}y_1^{2j}z_3t_1y_1^{2k-1}z_1^{2k-1} \\
L_3 &=& -4jz_2y_1z_1^{2j-1}y_1^{2j-1}t_1z_2y_1^{2k}z_3^{2k+2l}t_1^{2l} & L_9 &=& -4kz_2y_1z_1^{2k-1}y_1^{2k-1}t_1z_2y_1^{2l}z_3^{2l+2j}t_1^{2j} \\
L_4 &=& -4lz_3y_1t_1^{2j}z_2^{2j+2k}y_1^{2k}z_3t_1y_1^{2l-1}z_1^{2l-1} & L_{10} &=& -4jz_3y_1t_1^{2k}z_2^{2k+2l}y_1^{2l}z_3t_1y_1^{2j-1}z_1^{2j-1} \\
L_5 &=& -4kz_2y_1z_1^{2k-1}y_1^{2k-1}t_1z_2y_1^{2j}z_3^{2j+2l}t_1^{2l} & L_{11} &=& -4lz_2y_1z_1^{2l-1}y_1^{2l-1}t_1z_2y_1^{2k}z_3^{2k+2j}t_1^{2j} \\
L_6 &=& -4lz_3y_1t_1^{2k}z_2^{2k+2j}y_1^{2j}z_3t_1y_1^{2l-1}z_1^{2l-1} & L_{12} &=& -4jz_3y_1t_1^{2l}z_2^{2l+2k}y_1^{2k}z_3t_1y_1^{2j-1}z_1^{2j-1}.
\end{array}$$

None of these terms is colinear to $M_{m,p}$, because of the exponent of t_1 which is always too large.

• If $m+p+q = j+l+k$ with $1 \leq q \leq p \leq m$ and $(m,\, p,\, q) \neq (j,\, k,\, l)$, then $M_{m,p,q} = z_1 t_1 y_1^{2m+2q} z_3^{2m+2p} z_2^2 t_1^{2p} z_1^{2q-2}$ does not appear in $E_3'(P)$. In fact,
$\triangleright$ If $m + p > j + k$, then z_3 never appears in $E_3'(P)$ with the exponent $2m + 2p$.
$\triangleright$ If $m + p \leq j + k$, we write the terms of $E_3'(P)$ which have y_1 with exponent $2m + 2q$ and z_2 with exponent 2, by setting $C_n^\delta = 0$ if $\delta \notin [\![0,\, 2k]\!]$:

$$\begin{array}{lcl|lcl}
L_1 &=& -4jz_2y_1z_1^{2j-1}y_1^{2j-1}t_1z_2y_1^{2l}z_3^{2l+2k}t_1^{2k} & L_7 &=& -4lz_2y_1z_1^{2l-1}y_1^{2l-1}t_1z_2y_1^{2j}z_3^{2j+2k}t_1^{2k} \\
L_2 &=& -2C_{2k}^{\alpha}z_3y_1t_1^{2j}z_2^{2j+2l}y_1^{2l}z_3^{2k-\alpha}t_1^{2k-\alpha}y_1^{\alpha}z_1^{\alpha} & L_8 &=& -2C_{2k}^{\alpha}z_3y_1t_1^{2l}z_2^{2l+2j}y_1^{2j}z_3^{2k-\alpha}t_1^{2k-\alpha}y_1^{\alpha}z_1^{\alpha} \\
L_3 &=& -4jz_2y_1z_1^{2j-1}y_1^{2j-1}t_1z_2y_1^{2k}z_3^{2k+2l}t_1^{2l} & L_9 &=& -4kz_2y_1z_1^{2k-1}y_1^{2k-1}t_1z_2y_1^{2l}z_3^{2l+2j}t_1^{2j} \\
L_4 &=& -2C_{2l}^{\beta}z_3y_1t_1^{2j}z_2^{2j+2k}y_1^{2k}z_3^{2l-\beta}t_1^{2l-\beta}y_1^{\beta}z_1^{\beta} & L_{10} &=& -2C_{2j}^{\gamma}z_3y_1t_1^{2k}z_2^{2k+2l}y_1^{2l}z_3^{2j-\gamma}t_1^{2j-\gamma}y_1^{\gamma}z_1^{\gamma} \\
L_5 &=& -4kz_2y_1z_1^{2k-1}y_1^{2k-1}t_1z_2y_1^{2j}z_3^{2j+2l}t_1^{2l} & L_{11} &=& -4lz_2y_1z_1^{2l-1}y_1^{2l-1}t_1z_2y_1^{2k}z_3^{2k+2j}t_1^{2j} \\
L_6 &=& -2C_{2l}^{\beta}z_3y_1t_1^{2k}z_2^{2k+2j}y_1^{2j}z_3^{2l-\beta}t_1^{2l-\beta}y_1^{\beta}z_1^{\beta} & L_{12} &=& -2C_{2j}^{\gamma}z_3y_1t_1^{2l}z_2^{2l+2k}y_1^{2k}z_3^{2j-\gamma}t_1^{2j-\gamma}y_1^{\gamma}z_1^{\gamma},
\end{array}$$

with $\alpha \in [\![0,\, 2k]\!]$, $\beta \in [\![0,\, 2l]\!]$, $\gamma \in [\![0,\, 2j]\!]$ and $\alpha,\ \beta,\ \gamma$ odd.
None of these terms is colinear to $M_{m,p}$, because for each of the twelve terms, the equality of the multi-degrees gives a linear system which leads to an absurdity. For example, if L_1 is colinear to $M_{m,p}$, we have
$$\left\{\begin{array}{rcl} 2m+2q &=& 2j+2l \\ 2m+2p &=& 2l+2k \\ 2p+1 &=& 1+2k \\ 2q-1 &=& 2j-1 \end{array}\right.$$
, hence $p = k,\ q = j,\ m = l$, which is absurd by hypothesis.

Case 4. $P = R_3(X^{2i+1}Y^{2j+1}Z^{2k+1})$ (with $0 \leq i < j < k$) :
In this case, because of the degree, the only monomial for which we must show that it does not appear in $E_3'(P)$ is $\widetilde{M}_{m,p,q}$ with $m + p + q = j + k + i$, $0 < q < p < m$ and $(m,\, p,\, q) \neq (k,\, j,\, i)$. The study is the same as the one made in the third point of case 3.

So we have proved the following result :

Proposition 36
The dimension of the $0-th$ *space of Poisson homology of* B_3 *is* 3, *i.e.* $\boxed{\dim(HP_0(B_3)) = 3}$.
This dimension coincides with $\dim(HH_0(B_3))$.

As for the vectors of highest weight 0 of degree congruent to 2 modulo 4, we obtain recursively a formula which enables us to express them explicitly as sums of brackets : this is the subject of proposition 37.

Proposition 37
The vectors of highest weight 0 *of degree congruent to* 2 *modulo* 4 *are sums of brackets.*

Proof :
• We have the formula
$$\{R_3(X^{2p}),\ R_3(X^{2q})\} = \frac{8}{3}pq\, R_3\big(X^{2p-1}Y^{2q-1}Z\big) \tag{23}$$
Indeed,

$$
\begin{array}{rcl}
\{R_3(X^{2p}),\ R_3(X^{2q})\} & = & \dfrac{1}{|\mathfrak{S}_3|}\{\sum_{\sigma\in\mathfrak{S}_3}(x_{\sigma 1}y_{\sigma 2}-y_{\sigma 1}x_{\sigma 2})^{2p},\ \sum_{\tau\in\mathfrak{S}_3}(x_{\tau 1}y_{\tau 2}-y_{\tau 1}x_{\tau 2})^{2q}\} \\
& = & \dfrac{1}{9}\sum_{\sigma,\ \tau\in\langle(1,2,3)\rangle}\{(x_{\sigma 1}y_{\sigma 2}-y_{\sigma 1}x_{\sigma 2})^{2p},\ (x_{\tau 1}y_{\tau 2}-y_{\tau 1}x_{\tau 2})^{2q}\} \\
& = & \dfrac{4}{9}pq\sum_{\sigma,\ \tau\in\langle(1,2,3)\rangle}(x_{\sigma 1}y_{\sigma 2}-y_{\sigma 1}x_{\sigma 2})^{2p-1}(x_{\tau 1}y_{\tau 2}-y_{\tau 1}x_{\tau 2})^{2q-1}\{x_{\sigma 1}y_{\sigma 2}-y_{\sigma 1}x_{\sigma 2},\ x_{\tau 1}y_{\tau 2}-y_{\tau 1}x_{\tau 2}\} \\
& = & \dfrac{4}{9}pq\sum_{\tau\in\mathfrak{S}_3}\sigma\cdot(x_1y_2-y_1x_2)^{2p-1}(x_2y_3-y_2x_3)^{2q-1}(x_3y_1-y_3x_1) \\
& = & \dfrac{8}{3}pq\,R_3\big((x_1y_2-y_1x_2)^{2p-1}(x_2y_3-y_2x_3)^{2q-1}(x_3y_1-y_3x_1)\big),
\end{array}
$$

where the 4−th equality results from the table

$\{\cdot\}$	X	Y	Z
X	0	Z	$-Y$
Y	$-Z$	0	X
Z	Y	$-X$	0

.

We also have

$$
R_3\big(X^{2p-1}Y^{2q-1}Z\big)R_3\big(X^{2r}\big)=\frac{1}{3}\Big[R_3\big(X^{2p+2r-1}Y^{2q-1}Z\big)+R_3\big(X^{2p-1}Y^{2q+2r-1}Z\big)+R_3\big(X^{2p-1}Y^{2q-1}Z^{2r+1}\big)\Big] \tag{24}
$$

So, according to Leibniz property and formula (23), we have

$$
\begin{array}{rcl}
\{R_3(X^{2p}),\ R_3(X^{2q})R_3(X^{2r})\} & = & \{R_3(X^{2p}),\ R_3(X^{2q})\}R_3(X^{2r})+\{R_3(X^{2p}),\ R_3(X^{2r})\}R_3(X^{2q}) \\
& = & \frac{8}{3}pq\,R_3\big(X^{2p-1}Y^{2q-1}Z\big)R_3\big(X^{2r}\big)+\frac{8}{3}pr\,R_3\big(X^{2p-1}Y^{2r-1}Z\big)R_3\big(X^{2q}\big)
\end{array} \tag{25}
$$

• We deduce

$$
\begin{array}{rl}
& R_3(X^{2p-1}Y^{2q-1}Z^{2r+1}) \\
= & 3R_3(X^{2p-1}Y^{2q-1}Z)R_3(X^{2r})-R_3(X^{2p+2r-1}Y^{2q-1}Z)-R_3(X^{2p-1}Y^{2q+2r-1}Z) \\
= & \frac{9}{8pq}\{R_3(X^{2p}),\ R_3(X^{2q})R_3(X^{2r})\}-\frac{3r}{q}R_3(X^{2p-1}Y^{2q-1}Z)R_3(X^{2q})-R_3(X^{2p+2r-1}Y^{2q-1}Z)-R_3(X^{2p-1}Y^{2q+2r-1}Z) \\
= & \frac{9}{8pq}\{R_3(X^{2p}),\ R_3(X^{2q})R_3(X^{2r})\}-\frac{r}{q}\Big[R_3(X^{2p+2q-1}Y^{2r-1}Z)+R_3(X^{2p-1}Y^{2r+2q-1}Z)+R_3(X^{2p-1}Y^{2r-1}Z^{2q+1})\Big] \\
& -R_3(X^{2p+2r-1}Y^{2q-1}Z)-R_3(X^{2p-1}Y^{2q+2r-1}Z),
\end{array} \tag{26}
$$

where the equalities 1, 2, 3 result respectively from formulae (24), (25), (24).
• Then we proceed by recurrence on r : we show that for every $r\geq 1$ fixed, for every $(p,\,q)\in(\mathbb{N}^*)^2$ such that $r<q<p$, the element $R_3(X^{2p-1}Y^{2q-1}Z^{2r-1})$ is a sum of brackets : in fact, the case $r=1$ comes from (23) while the way from r to $r+1$ results from (26).■

3.4 Study of D_3

The result for D_3 may be deduced from the result for B_3.
In fact, the vectors of highest weight 0 are the same as the ones of B_3 (this results for example from Proposition 16), and they are given by remark 34.
The element 1 is not a bracket, according to Proposition 4.
Equation (17) is the same as for B_3. This shows that the solutions of equation (11) are to be searched in the vector space $\langle R_3(X^2),\ R_3(X^2Y^2)\rangle$.
But these two polynomials have distinct degrees and we verify with the help of Maple that none of them is a solution of equation (11) : see section 4.

So we have the following result :

Proposition 38
The dimension of the 0−*th space of Poisson homology of* D_3 *is* 1*, i.e.* $\boxed{\dim(HP_0(D_3))=1}$.
This dimension coincides with $\dim(HH_0(D_3))$.

Remark 39
Propositions 31, 32, 36 and 38 show that the conjecture of J. Alev holds in the cases B_2, $D_2=A_1\times A_1$, B_3 *and* $D_3=A_3$.

Remark 40
J. Alev conjectured that the equality $\dim\,HP_0(W)=\dim\,HH_0(W)$ *holds not only for the Weyl groups of semi-simple finite-dimensional Lie algebras, but also for the more general case of wreath products of she form* $W=$

$\Gamma \smile \mathfrak{S}_n$, *where* Γ *is a finite subgroup of* $\mathbf{SL}_2\mathbb{C}$. *In particular :*
• *If* $\Gamma = A_1$, *we have* $W = \mathfrak{S}_n$. *In this case, an explicit calculation shows that* $\dim HP_0(W) = 0 = \dim HH_0(W)$.
• *If* $\Gamma = A_2$, *we have* $W \simeq B_n$. *The cases* $n = 2$ *and* $n = 3$ *have been verified ([AF06] and the present article). For the case* $n \geq 4$, *this is still a conjecture !*

4 Formal computations

We collect in this section some verifications which are carried out with Maple.

4.1 Definitions

• The function Image calculates the image of the polynomial $P \in \mathbb{A}[\mathbf{X}]$ by the matrix $J \in \mathbf{Gl}_n\mathbb{C}$, according to the diagonal action, with $\mathbf{X} = (\mathbf{x},\ \mathbf{y})$, and $\mathbb{A} = \mathbb{C}$ or $\mathbb{A} = \mathbb{C}[\mathbf{z},\ \mathbf{t}]$.

```
Image:=proc(P,X::list,J) local n: n:=nops(X)/2:
subs({seq(X[i]=add(J[i,j]*X[j],j=1..n),i=1..n),seq(X[i]=add(J[i-n,j-n]*X[j],
j=n+1..2*n),i=n+1..2*n)},P); end proc:
```

• The function Reynolds calculates the image by the Reynolds operator associated to the group $W \subset \mathbf{Gl}_n\mathbb{C}$ of the polynomial $P \in \mathbb{A}[\mathbf{X}]$, according to the diagonal action.

```
Reynolds:=proc(P,X::list,W::list) local card,n: card:=nops(W):
n:=nops(X)/2: 1/card*add(Image(P,X,W[j]),j=1..card); end proc:
```

• The function kron gives the Kronecker symbol of $(i,\ j)$.

```
kron:=proc(i,j) if i=j then 1 else 0 fi; end proc:
```

• The function repr gets a permutation $\sigma \in \mathfrak{S}_n$, written in the shape of a list $[\sigma(1), \ldots,\ \sigma(n)]$, and gives the matrix of size n and of general term $\delta_{i,\sigma(j)}$, i.e. the permutation matrix associated to σ.

```
repr:=proc(sigma) local n: n:=nops(sigma):
Matrix(n,(i,j)->kron(i,sigma[j])); end proc:
```

• Definition of the Weyl groups of B_2 and D_2 :

```
t12:=<<0,1>|<1,0>>: s1:=<<-1,0>|<0,1>>: s2:=<<1,0>|<0,-1>>:
s12:=<<-1,0>|<0,-1>>:
B2:=[seq(seq(seq(t12^i.s1^j.s2^k,i=0..1),j=0..1),k=0..1)]:
D2:=[seq(seq(t12^i.s12^j,i=0..1),j=0..1)]:
```

• Definition of the Weyl group of B_3 :

```
with(group): S3gr:=permgroup(3,{[[1,2]],[[1,2,3]]});
S3grbis:=[op(elements(S3gr))]:
S3grliste:=map(x->convert(x,'permlist',3),S3grbis):
S3:=map(repr,S3grliste): s1:=<<-1,0,0>|<0,1,0>|<0,0,1>>:
s2:=<<1,0,0>|<0,-1,0>|<0,0,1>>: s3:=<<1,0,0>|<0,1,0>|<0,0,-1>>:
B3:=[seq(seq(seq(seq(S3[i].s1^j.s2^k.s3^l,i=1..6),j=0..1),k=0..1),l=0..1)]:
```

• Definition of the Weyl group of D_3 :

```
s12:=<<-1,0,0>|<0,-1,0>|<0,0,1>>:s23:=<<1,0,0>|<0,-1,0>|<0,0,-1>>:
D3:=[seq(seq(seq(S3[i].s12^j.s23^k,i=1..6),j=0..1),k=0..1)]:
```

4.2 Verification of the calculations of propositions 31 and 32

The following calculations enable us to verify that $E_2(R_2(X^2)) = 0$ for B_2 and $E_2(R_2(X^2)) \neq 0$ for D_2.

```
ind:=[seq(x[j],j=1..2),seq(y[j],j=1..2)]:
```

```
uB:=Reynolds((x[1]*y[2]-y[1]*x[2])^2,ind,B2);
vB:=add(z[j]*y[j]-t[j]*x[j],j=1..2)*subs({seq(x[j]=x[j]+z[j],j=1..2),
seq(y[j]=y[j]+t[j],j=1..2)},uB):
equaB:=Reynolds(vB,ind,B2):equaBd:=expand(equaB);
```

```
uD:=Reynolds((x[1]*y[2]-y[1]*x[2])^2,ind,D2);
vD:=add(z[j]*y[j]-t[j]*x[j],j=1..2)*subs({seq(x[j]=x[j]+z[j],j=1..2),
seq(y[j]=y[j]+t[j],j=1..2)},uD):
equaD:=Reynolds(vD,ind,D2):equaDd:=expand(equaD);
```

4.3 Verification of the identities of Proposition 35

The following calculations prove the identities $E_3(R_3(X^2)) = 0$ and $E_3(R_3(X^2Y^2)) = 0$.

```
ind:=[seq(x[j],j=1..3),seq(y[j],j=1..3)]:
u1:=Reynolds((x[1]*y[2]-y[1]*x[2])^2,ind,B3):
v1:=add(z[j]*y[j]-t[j]*x[j],j=1..3)*subs({seq(x[j]=x[j]+z[j],j=1..3),
seq(y[j]=y[j]+t[j],j=1..3)},u1):
equa1:=Reynolds(v1,ind,B3):equad1:=expand(equa1);
u2:=Reynolds((x[1]*y[2]-y[1]*x[2])^2*(x[2]*y[3]-y[2]*x[3])^2,ind,B3):
v2:=add(z[j]*y[j]-t[j]*x[j],j=1..3)*subs({seq(x[j]=x[j]+z[j],j=1..3),
seq(y[j]=y[j]+t[j],j=1..3)},u2):
equa2:=Reynolds(v2,ind,B3):equad2:=expand(equa2);
```

4.4 Verification of the calculations of Proposition 38

The following calculations show that the polynomials $R_3(X^2)$ and $R_3(X^2Y^2)$ are not solutions of equation (11).

```
ind:=[seq(x[j],j=1..3),seq(y[j],j=1..3)]:
u1:=Reynolds((x[1]*y[2]-y[1]*x[2])^2,ind,D3):
v1:=add(z[j]*y[j]-t[j]*x[j],j=1..3)*subs({seq(x[j]=x[j]+z[j],j=1..3),
seq(y[j]=y[j]+t[j],j=1..3)},u1):
equa1:=Reynolds(v1,ind,D3):equad1:=expand(equa1): nops(equad1);
u2:=Reynolds((x[1]*y[2]-y[1]*x[2])^2*(x[2]*y[3]-y[2]*x[3])^2,ind,D3):
v2:=add(z[j]*y[j]-t[j]*x[j],j=1..3)*subs({seq(x[j]=x[j]+z[j],j=1..3),
seq(y[j]=y[j]+t[j],j=1..3)},u2):
equa2:=Reynolds(v2,ind,D3):equad2:=expand(equa2): nops(equad2);
```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